Jackson 4.2 Homework Problem Solution

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PROBLEM:

A point dipole with dipole moment **p** is located at the point $\mathbf{x}_0$. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_0)$$

SOLUTION:

The potential due to a dipole is:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}$$

The potential due to a charge density is:

$$\Phi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

so that:

$$\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

$$\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3} = \int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

Use the relation proved earlier that $\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3} = \nabla_{x_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right)$ where **r** is the separation vector

$$\mathbf{p}\cdot\nabla_{x_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) = \int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

Add to the left an integral and a Dirac delta:

$$\int\delta(\mathbf{x}'-\mathbf{x}_0)\,\mathbf{p}\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d\mathbf{x}' = \int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

Use integration by parts

$$-\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{p}\cdot\nabla'\left(\delta(\mathbf{x}'-\mathbf{x}_0)\right)d\mathbf{x}'=\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

Shrink down the integrals until it must be true at every point, so that the integrands must match.

$$\rho(\mathbf{x}')=-\mathbf{p}\cdot\nabla'\left(\delta(\mathbf{x}'-\mathbf{x}_0)\right)$$

Relabel the primed variables to be unprimed:

$$\boxed{\rho_{\text{eff}}(\mathbf{x})=-\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_0)}$$

We can also go other way in the same manner, putting this into Coulomb's law and ending up with the dipole potential.

The energy of a dipole in an external field is $W=[-\mathbf{p}\cdot\nabla\Phi]_{\mathbf{x}=\mathbf{x}_0}$

The energy in general of a charge distribution in an external field is $W=\int\rho(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}$

So that:

$$[-\mathbf{p}\cdot\nabla\Phi(\mathbf{x})]_{\mathbf{x}=\mathbf{x}_0}=\int\rho(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}$$

$$-\int\delta(\mathbf{x}-\mathbf{x}_0)\mathbf{p}\cdot\nabla\Phi(\mathbf{x})d\mathbf{x}=\int\rho(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}$$

Applying integration by parts:

$$-\int\Phi(\mathbf{x})\mathbf{p}\cdot\nabla\left(\delta(\mathbf{x}-\mathbf{x}_0)\right)d\mathbf{x}=\int\rho(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}$$

Equating integrands:

$$\boxed{\rho_{\text{eff}}(\mathbf{x})=-\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_0)}$$