Jackson 4.1 Homework Problem Solution

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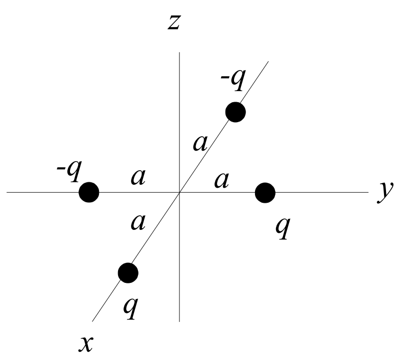
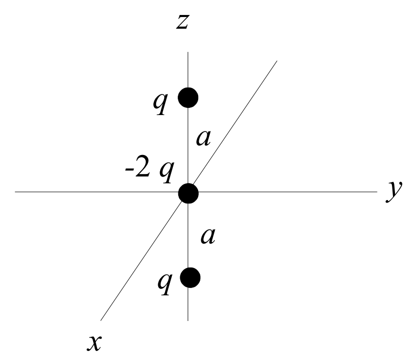
PROBLEM:

Calculate the multipole moments q_{lm} of the charge distributions shown as parts a and b. Try to obtain results for the non-vanishing moments valid for all l, but in each case find the first two sets of non-vanishing moments at the very least.

(a)



(b)



(c) For the charge distribution of the second set b) write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the x-y plane as a function of distance from the origin for distances greater than a.

(d) Calculate directly from Coulomb's law the exact potential for b) in the x-y plane. Plot it as a function of distance and compare with the result found in part c).

SOLUTION:

(a) The charge density is written down in spherical coordinates as:

$$\rho=\frac{q}{a^2}\delta(r-a)\delta(\cos\theta)\left[\delta(\phi)-\delta(\phi-3\pi/2)-\delta(\phi-\pi)+\delta(\phi-\pi/2)\right]$$

Plug this into the multipole moments definition and evaluate:

$$q_{lm}=\int Y^*_{lm}(\theta',\phi')r'^l\rho(\mathbf{x}')d\mathbf{x}'$$

$$q_{lm}=\frac{q}{a^2}\int Y^*_{lm}(\theta',\phi')r'^l\delta(r'-a)\delta(\cos\theta')\left[\delta(\phi')-\delta(\phi'-3\pi/2)-\delta(\phi'-\pi)+\delta(\phi'-\pi/2)\right]d\mathbf{x}'$$

$$q_{lm}=q\,a^l\int_0^{2\pi}\int_0^{\pi} Y^*_{lm}(\theta',\phi')\delta(\cos\theta')\left[\delta(\phi')-\delta(\phi'-3\pi/2)-\delta(\phi'-\pi)+\delta(\phi'-\pi/2)\right]\sin\theta' d\theta' d\phi'$$

$$q_{lm}=q\,a^l\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(0)\int_0^{2\pi}e^{-im\phi'}[\delta(\phi')-\delta(\phi'-3\pi/2)-\delta(\phi'-\pi)+\delta(\phi'-\pi/2)]\,d\phi'$$

$$q_{lm}=q\,a^l\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1-i^m-(-1)^m+(-1)^m i^m\right]$$

$$q_{lm}=q\,a^l\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(0)(1-(-1)^m)(1-i^m)$$

$$\boxed{q_{lm}=2q\,a^l\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(0)(1-i^m)}$$ if m is odd, $q_{lm}=0$ if m is even

This is the solution valid for all l.
Let us write out the first few multipoles explicitly to see what this means.
The monopole moment is:

$$q_{00}=0 \quad \text{because } m \text{ is even}$$

This makes sense because the total charge is zero.
The dipole moments are:

$$q_{1,-1}=q\,a\sqrt{\frac{3}{2\pi}}(1+i)$$

$$q_{1,0}=0$$

$$q_{11}=q\,a\sqrt{\frac{3}{2\pi}}(-1+i)$$

Put these together into the Cartesian dipole moment vector:

$$\mathbf{p}=p_x\hat{\mathbf{i}}+p_y\hat{\mathbf{j}}+p_z\hat{\mathbf{k}}$$

$$\mathbf{p}=\sqrt{\frac{2\pi}{3}}\left((q_{1,-1}-q_{11})\hat{\mathbf{i}}-i(q_{1,-1}+q_{11})\hat{\mathbf{j}}+\sqrt{2}q_{10}\hat{\mathbf{k}}\right)$$

$$\mathbf{p}=2q\,a\left(\hat{\mathbf{i}}+\hat{\mathbf{j}}\right)$$

The quadrupole moments are:

$$q_{2,-2}=0$$

$$q_{2,-1}=0$$

$$q_{2,0}=0$$

$$q_{21}=0$$

$$q_{2,2}=0$$

The problem asks us for the first two sets of non-vanishing moments, so we have to keep going. The non-zero octupole moments are:

$$q_{3,-3}=qa^3\sqrt{\frac{35}{16\pi}}(1-i)$$

$$q_{3,-1}=qa^3\sqrt{\frac{21}{16\pi}}(-1-i)$$

$$q_{31}=q\,a^3\sqrt{\frac{21}{16\pi}}(1-i)$$

$$q_{33}=q\,a^3\sqrt{\frac{35}{16\pi}}(-1-i)$$

(b) The charge density is written down in spherical coordinates as:

$$\rho=\frac{q}{2\pi a^2}\delta(r-a)\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]-\frac{2q}{4\pi r^2}\delta(r)$$

Plug this into the multipole moments definition and evaluate:

$$q_{lm}=\int Y_{lm}^*(\theta',\phi')r'^l\rho(\mathbf{x}')d\mathbf{x}'$$

$$q_{lm}=\int Y_{lm}^*(\theta',\phi')r'^l\left[\frac{q}{2\pi a^2}\delta(r'-a)\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right]-\frac{2q}{4\pi r'^2}\delta(r')\right]d\mathbf{x}'$$

$$q_{lm}=\frac{q}{2\pi}\int_0^{2\pi}\int_0^{\pi}Y_{lm}^*(\theta',\phi')a^l\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right]\sin\theta'd\theta'd\phi'$$
$$-\frac{2q}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}Y_{lm}^*(\theta',\phi')r'^l\delta(r')\sin\theta'dr'd\theta'd\phi'$$

Expand the spherical harmonics and try to do the integral over the azimuthal coordinate. We find that it vanishes except for when $m = 0$. This makes sense because the problem has azimuthal symmetry. In the second integral, all terms disappear except for $l = 0$.

$$q_{lm} = \delta_{m,0}\left[q a^l \sqrt{\frac{2l+1}{4\pi}} \left[1+(-1)^l\right] - \delta_{l,0}\, q \sqrt{\frac{1}{\pi}} \right]$$

This becomes more clear if we break this into different cases, including the simplest monopoles first:

$$q_{00} = 0$$

$$q_{10} = q_{1,-1} = q_{11} = 0$$

$$q_{20} = \sqrt{\frac{5}{\pi}}\, q a^2 \qquad q_{2,-2} = q_{2,-1} = q_{2,1} = q_{2,2} = 0$$

$$q_{3,-3} = q_{3,-2} = q_{3,-1} = q_{3,0} = q_{3,1} = q_{3,2} = q_{3,3} = 0$$

$$q_{40} = \sqrt{\frac{9}{\pi}}\, q a^4$$

and all higher terms can be expressed as:

$\boxed{q_{lm} = \sqrt{\frac{2l+1}{\pi}}\, q a^l}$ if $m = 0$ and $l =$ even, $q_{lm} = 0$ otherwise

(c) For the charge distribution of the second set b) write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the x-y plane as a function of distance from the origin for distances greater than a.

The multipole expansion of the potential is:

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} r^{-l-1} Y_{lm}(\theta, \phi)$$

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{4\pi}{5} q_{20} \frac{1}{r^3} Y_{20}(\theta,\phi) + \frac{4\pi}{9} q_{40} \frac{1}{r^5} Y_{40}(\theta,\phi) + \sum_{l=6,\text{even}}^{\infty} \frac{4\pi}{2l+1} q_{l,0} \frac{Y_{l0}(\theta,\phi)}{r^{l+1}} \right]$$

$$\Phi = \frac{q}{4\pi\epsilon_0 a}\left[\left(\frac{a}{r}\right)^3 (3\cos^2\theta - 1) + \left(\frac{a}{r}\right)^5 \frac{1}{4}(35\cos^4\theta - 20\cos^2\theta + 3) + 2 \sum_{l=6,\text{ even}}^{\infty} \left(\frac{a}{r}\right)^{l+1} P_l(\theta,\phi) \right]$$

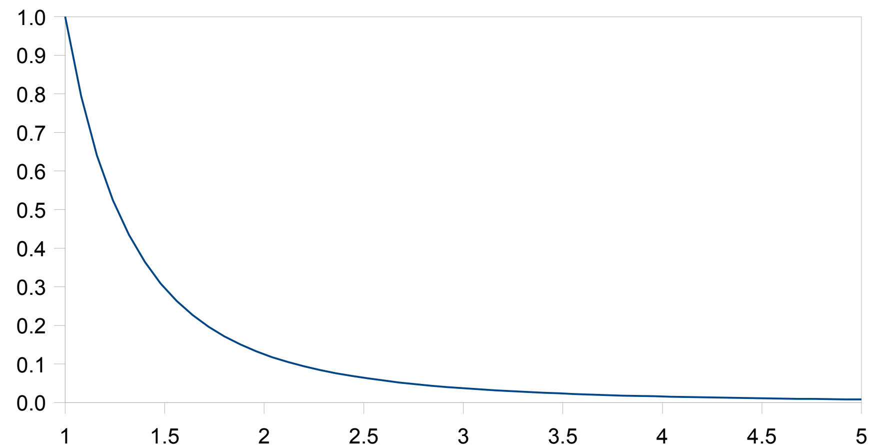
Now if r is much greater than a, then (a/r) is much less than one and (a/r) raised to higher powers is even smaller, so that we only need to keep the first term.

$$\Phi = \frac{q}{4\pi\epsilon_0 a}\left(\frac{a}{r}\right)^3 (3\cos^2\theta - 1)$$

In the *x*-*y* plane this becomes:

$$\Phi = \frac{-q}{4\pi\epsilon_0 a}\left(\frac{a}{r}\right)^3$$

We can plot the potential in units of ($-q/4\pi\varepsilon_0 a$) and the distance in unit of a:



(d) Calculate directly from Coulomb's law the exact potential for b) in the x-y plane. Plot it as a function of distance and compare with the result found in part c).

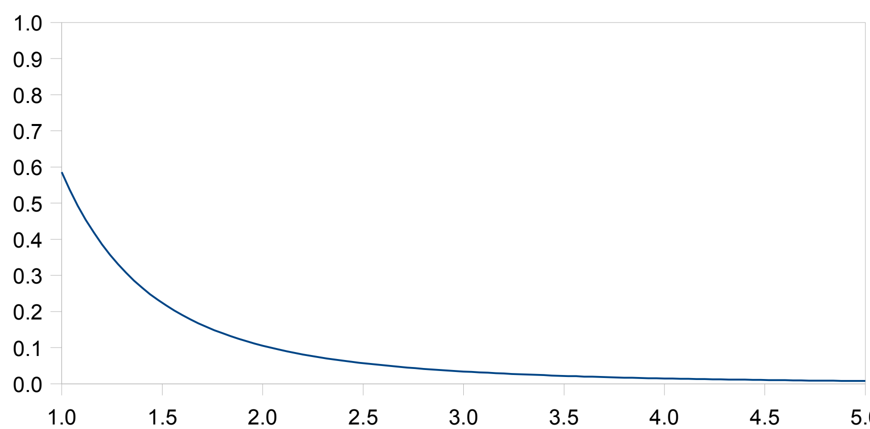
We have three points charge and can write out an exact solution for the point charges:

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{r}+a\hat{\mathbf{z}}|} + \frac{1}{|\mathbf{r}-a\hat{\mathbf{z}}|} - \frac{2}{|\mathbf{r}|}\right]$$

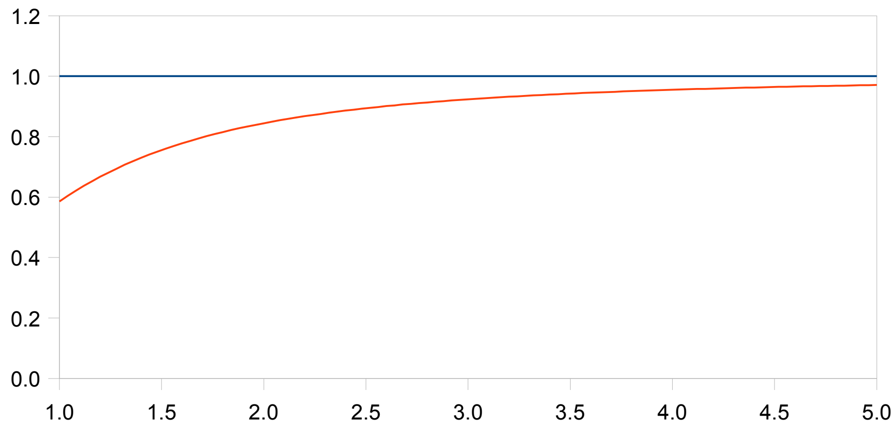
In the *x*-*y* plane this becomes:

$$\Phi = \frac{2q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2}} - \frac{1}{r}\right]$$

$$\Phi = \frac{-q}{4\pi\epsilon_0 a}\left[\frac{2}{(r/a)} - \frac{2}{\sqrt{(r/a)^2+1}}\right]$$



It becomes apparent now that the first term in the multipole expansion is a good approximation to as close as $r = 2a$, but becomes inaccurate closer than that.
Dividing out the asymptotic form (a^3/r^3) lets us compare them for clearly:


1.2
1.0
0.8
0.6
0.4
0.2
0.0
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0