



C. S. BAIRD

Jackson 4.13 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell



PROBLEM:

Two long, coaxial, cylindrical conducting surfaces of radii a and b are lowered vertically into a liquid dielectric. If the liquid rises an average height h between the electrodes when a potential difference V is established between them, show that the susceptibility of the liquid is

$$\chi_e = \frac{(b^2 - a^2) \rho g h \ln(b/a)}{\epsilon_0 V^2}$$

where ρ is the density of the liquid, g is the acceleration due to gravity, and the susceptibility of air is neglected.

SOLUTION:

There are two regions, the one between the cylinders with air and the one between the cylinders with the liquid. Apart from different permittivities, the \mathbf{D} fields in both regions will be identical. Place the axis of the cylinder on the z axis. Because the cylinder is long, we can ignore the z dimension and we just need to solve the Laplace equation in polar coordinates.

$$\nabla^2 \Phi = 0$$

The general solution is:

$$\Phi(\rho, \phi) = (a_0 + b_0 \ln \rho)(1 + B_0 \phi) + \sum_{\nu \neq 0} (a_\nu \rho^\nu + b_\nu \rho^{-\nu})(e^{i\nu\phi} + B_\nu e^{-i\nu\phi})$$

This problem has azimuthal symmetry so that the solution must have azimuthal symmetry. This makes many terms in the general solution go away, leaving:

$$\Phi(\rho, \phi) = a_0 + b_0 \ln \rho$$

The problem just states that there is a potential difference between the cylinders. Let us set the inner one at zero and the outer one at V . Apply the inner boundary condition:

$$\Phi(\rho = a) = 0$$

$$0 = a_0 + b_0 \ln a$$

$$a_0 = -b_0 \ln a$$

The solution is now:

$$\Phi(\rho, \phi) = b_0 \ln(\rho/a)$$

Apply the boundary condition at the outer cylinder:

$$\Phi(\rho=b)=V$$

$$V=b_0 \ln(b/a)$$

$$b_0 = \frac{V}{\ln(b/a)}$$

The solution is now:

$$\Phi(\rho, \phi) = V \frac{\ln(\rho/a)}{\ln(b/a)}$$

The electric field is therefore:

$$\mathbf{E} = -\nabla \Phi$$

$$\mathbf{E} = -\hat{\rho} V \frac{1}{\ln(b/a)} \frac{1}{\rho}$$

Before lowering the cylinders of length l , the region is completely filled with air and the fields have some total potential energy W_{before} . After lowering the cylinders and reaching equilibrium, the presence of the dielectric means we have a new total potential energy in the fields W_{after} which is higher than the original energy. Conservation of energy is not violated because maintaining a constant potential difference V before and after means that we externally provide energy to the system to maintain the potential as the system changes. This is where the field gets its energy. The change in field potential energy ΔW will therefore depend on the properties of the liquid introduced.

$$\Delta W = W_{\text{after}} - W_{\text{before}}$$

$$\Delta W = \frac{1}{2} \int \mathbf{E}_{\text{after}} \cdot \mathbf{D}_{\text{after}} d^3 \mathbf{x} - \frac{1}{2} \int \mathbf{E}_{\text{before}} \cdot \mathbf{D}_{\text{before}} d^3 \mathbf{x}$$

$$\Delta W = \left[\frac{1}{2} (L-h) \epsilon_0 \int E_{\text{air}}^2 d^2 \mathbf{x} + \frac{1}{2} \epsilon_0 (1 + \chi_e) h \int E_{\text{liquid}}^2 d^2 \mathbf{x} \right] - \left[\frac{1}{2} L \epsilon_0 \int E_{\text{air}}^2 d^2 \mathbf{x} \right]$$

$$\Delta W = -\frac{1}{2} h \epsilon_0 \int E_{\text{air}}^2 d^2 \mathbf{x} + \frac{1}{2} \epsilon_0 (1 + \chi_e) h \int E_{\text{liquid}}^2 d^2 \mathbf{x}$$

The \mathbf{E} field is the same in both regions, so that we have:

$$\Delta W = \frac{h \epsilon_0 \chi_e}{2} \int E^2 d^2 \mathbf{x}$$

$$\Delta W = \frac{h \epsilon_0 \chi_e}{2} 2\pi \int_a^b E^2 \rho d\rho$$

$$\Delta W = \frac{h \epsilon_0 \chi_e}{2} 2\pi \int_a^b \left(V^2 \frac{1}{\ln^2(b/a)} \frac{1}{\rho^2} \right) \rho d\rho$$

$$\Delta W = \frac{\pi h \epsilon_0 \chi_e V^2}{\ln(b/a)}$$

This is the total potential energy gain in the fields that results from introducing a dielectric up to height h . The potential energy can be viewed as held in the fields, as we have just done, or as held in the particles. The particles can be viewed as gaining a total gravitational potential energy.

$$\Delta W = m g h$$

$$\Delta W = \rho h A g h$$

$$\Delta W = \rho \pi (b^2 - a^2) g h^2$$

Both methods are equivalent, so that:

$$\frac{\pi h \epsilon_0 \chi_e V^2}{\ln(b/a)} = \rho \pi (b^2 - a^2) g h^2$$

leading to:

$$\chi_e = \frac{(b^2 - a^2) \rho g h \ln(b/a)}{\epsilon_0 V^2}$$