Jackson 3.9 Homework Problem Solution

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PROBLEM:

A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

SOLUTION:

There is no charge present, so we seek to solve the Laplace equation in cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2}+\frac{\partial^2 \Phi}{\partial z^2}=0$$

Using the method of separation of variables, the general solution to this equation is:

$$\begin{aligned}\Phi(\rho,\phi,z)=&(A_{0,0}+B_{0,0}\ln\rho)(C_{0,0}+D_{0,0}\phi)(F_{0,0}+G_{0,0}z)\\&+\sum_{\nu\neq 0}(A_{\nu,0}\rho^{\nu}+B_{\nu,0}\rho^{-\nu})(C_{\nu,0}e^{i\nu\phi}+D_{\nu,0}e^{-i\nu\phi})(F_{\nu,0}+G_{\nu,0}z)\\&+\sum_{k\neq 0}(A_{0,k}J_0(k\rho)+B_{0,k}N_0(k\rho))(C_{0,k}+D_{0,k}\phi)(F_{0,k}e^{kz}+G_{0,k}e^{-kz})\\&+\sum_{\nu\neq 0}\sum_{k\neq 0}(A_{\nu,k}J_{\nu}(k\rho)+B_{\nu,k}N_{\nu}(k\rho))(C_{\nu,k}e^{i\nu\phi}+D_{\nu,k}e^{-i\nu\phi})(F_{\nu,k}e^{kz}+G_{\nu,k}e^{-kz})\end{aligned}$$

We seek a solution that must be valid on the full azimuthal sweep. This single-valued requirement leads to $D_{0,0}=0$, $D_{0,k}=0$, and $\nu=m$ where $m=0,1,2,3,\ldots$

$$\begin{aligned}\Phi(\rho,\phi,z)=&(A_{0,0}+B_{0,0}\ln\rho)(F_{0,0}+G_{0,0}z)\\&+\sum_{m\neq 0}(A_{m,0}\rho^{m}+B_{\nu,0}\rho^{-m})(C_{\nu,0}e^{im\phi}+D_{\nu,0}e^{-im\phi})(F_{m,0}+G_{m,0}z)\\&+\sum_{k\neq 0}(A_{0,k}J_0(k\rho)+B_{0,k}N_0(k\rho))(F_{0,k}e^{kz}+G_{0,k}e^{-kz})\\&+\sum_{m\neq 0}\sum_{k\neq 0}(A_{m,k}J_{m}(k\rho)+B_{m,k}N_{m}(k\rho))(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})(F_{\nu,k}e^{kz}+G_{m,k}e^{-kz})\end{aligned}$$

We also seek a solution that is valid along the z axis. The Neumann functions, the ρ^{-m} terms, and the $\ln \rho$ terms blow up on this axis, so their coefficients must be set to zero to keep the solution valid. This leads to:

$$\Phi(\rho,\phi,z)=(F_{0,0}+G_{0,0}z)$$
$$+\sum_{m\neq 0}\rho^m(C_{\nu,0}e^{im\phi}+D_{\nu,0}e^{-im\phi})(F_{m,0}+G_{m,0}z)$$
$$+\sum_{k\neq 0}J_0(k\rho)(F_{0,k}e^{kz}+G_{0,k}e^{-kz})$$
$$+\sum_{m\neq 0}\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})(F_{\nu,k}e^{kz}+G_{m,k}e^{-kz})$$

At this point, several terms can be combined:

$$\Phi(\rho,\phi,z)=\sum_m\rho^m(C_{\nu,0}e^{im\phi}+D_{\nu,0}e^{-im\phi})(F_{m,0}+G_{m,0}z)$$
$$+\sum_m\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})(F_{\nu,k}e^{kz}+G_{m,k}e^{-kz})$$

Apply the boundary condition $\Phi(z=0)=0$

$$0=\sum_m\rho^m(C_{\nu,0}e^{im\phi}+D_{\nu,0}e^{-im\phi})(F_{m,0})+\sum_m\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})(F_{\nu,k}+G_{m,k})$$

$$0=F_{m,0}$$

$$G_{m,k}=-F_{\nu,k}$$

This leads to:

$$\Phi(\rho,\phi,z)=\sum_m\rho^m(C_{\nu,0}e^{im\phi}+D_{\nu,0}e^{-im\phi})G_{m,0}z+\sum_m\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})\sinh(kz)$$

Apply the boundary condition $\Phi(z=L)=0$

$$0=\sum_m\rho^m(C_{\nu,0}e^{im\phi}+D_{\nu,0}e^{-im\phi})G_{m,0}L+\sum_m\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})\sinh(kL)$$

$$0=G_{m,0}$$

$$0=\sinh(kL)$$

$$k=i\frac{n\pi}{L}\ \text{where}\ n=1,2,\ldots$$

Note that the $n = 0$ case cannot be included, because it corresponds to $k = 0$, which was already handled separately and is not included in the second sum.
This leads to (using $\sinh(i\theta)=i\sin(\theta)$):

$$\Phi(\rho,\phi,z)=\sum_{m=0,1,2\ldots}\sum_{n=1,2\ldots}J_m\left(in\pi\frac{\rho}{L}\right)(C_{m,n}e^{im\phi}+D_{m,n}e^{-im\phi})\sin\left(n\pi\frac{z}{L}\right)$$

The Bessel functions of pure imaginary argument are defined to be modified Bessel functions:

$$J_\nu(ix)=i^\nu I_\nu(x)$$

$$\Phi(\rho,\phi,z)=\sum_{m=0,1,2...}\sum_{n=1,2...} i^m I_m\left(n\pi\frac{\rho}{L}\right)(C_{m,n}e^{im\phi}+D_{m,n}e^{-im\phi})\sin\left(n\pi\frac{z}{L}\right)$$

Apply the final boundary condition: $\Phi(\rho=b)=V(\phi,z)$

$$V(\phi,z)=\sum_{m=0,1,2...}\sum_{n=1,2...} i^m I_m\left(n\pi\frac{b}{L}\right)\sin\left(n\pi\frac{z}{L}\right)(C_{m,n}e^{im\phi}+D_{m,n}e^{-im\phi})$$

This is just a two-dimensional Fourier series. Multiply both side by a negative complex exponential and integrate:

$$\int_0^{2\pi} V(\phi,z)e^{-im'\phi}d\phi=\sum_{m=0,1,2...}\sum_{n=1,2...} i^m I_m\left(n\pi\frac{b}{L}\right)\sin\left(n\pi\frac{z}{L}\right)\left(C_{m,n}\int_0^{2\pi} e^{i(m-m')\phi}d\phi+D_{m,n}\int_0^{2\pi} e^{-i(m+m')\phi}d\phi\right)$$

Use the orthogonality of the complex exponentials: $\int_0^{2\pi} e^{i(k-k')x}dx=2\pi\,\delta_{k,k'}$

$$\int_0^{2\pi} V(\phi,z)e^{-im\phi}d\phi=\sum_{n=1,2...} i^m I_m\left(n\pi\frac{b}{L}\right)\sin\left(n\pi\frac{z}{L}\right)(C_{m,n}2\pi)$$

Multiply both sides by a sine function and integrate:

$$\int_0^L\int_0^{2\pi} V(\phi,z)e^{-im\phi}d\phi\sin\left(n'\pi\frac{z}{L}\right)dz=\sum_{n=1,2...} i^m I_m\left(n\pi\frac{b}{L}\right)\int_0^L \sin\left(n\pi\frac{z}{L}\right)\sin\left(n'\pi\frac{z}{L}\right)dz\,(C_{m,n}2\pi)$$

Use the orthogonality of the sine functions: $\int_0^L \sin\left(n\pi\frac{x}{L}\right)\sin\left(n'\pi\frac{x}{L}\right)dx=\frac{L}{2}\delta_{n,n'}$

$$\int_0^L\int_0^{2\pi} V(\phi,z)e^{-im\phi}d\phi\sin\left(n\pi\frac{z}{L}\right)dz=i^m I_m\left(n\pi\frac{b}{L}\right)\frac{L}{2}(C_{m,n}2\pi)$$

Solve for the coefficients:

$$C_{m,n}=\frac{1}{L\pi i^m I_m\left(n\pi\frac{b}{L}\right)}\int_0^L\int_0^{2\pi} V(\phi,z)e^{-im\phi}\sin\left(n\pi\frac{z}{L}\right)d\phi\,dz$$

The exact same process is repeated, this time starting by multiplying by a positive complex exponential

$$D_{m,n}=\frac{1}{L\pi i^m I_m\left(n\pi\frac{b}{L}\right)}\int_0^L\int_0^{2\pi} V(\phi,z)e^{im\phi}\sin\left(n\pi\frac{z}{L}\right)d\phi\,dz$$

After inserting these coefficients, the final solution becomes:

$$\boxed{\Phi(\rho,\phi,z)=\frac{1}{L\pi}\sum_{m=0,1,2...}^{\infty}\sum_{n=1,2...}^{\infty}\frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\left(C_{m,n}e^{im\phi}+C_{m,n}^{*}e^{-im\phi}\right)\sin\left(n\pi\frac{z}{L}\right)}$$

where $\boxed{C_{m,n}=\int_0^L\int_0^{2\pi}V(\phi,z)e^{-im\phi}\sin\left(n\pi\frac{z}{L}\right)d\phi\,dz}$

We can formulate this entirely in terms of real-valued parameters by expanding out the complex exponentials to find:

$$\boxed{\Phi(\rho,\phi,z)=\frac{1}{L\pi}\sum_{m=0,1,2...}^{\infty}\sum_{n=1,2...}^{\infty}\frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\sin\left(n\pi\frac{z}{L}\right)2\left[A_{m,n}\cos(m\phi)+B_{m,n}\sin(m\phi)\right]}$$

where $\boxed{A_{m,n}=\int_0^L\int_0^{2\pi}V(\phi',z')\cos(m\phi')\sin\left(\frac{n\pi z'}{L}\right)d\phi'\,dz'}$ and

$$\boxed{B_{m,n}=\int_0^L\int_0^{2\pi}V(\phi',z')\sin(m\phi')\sin\left(\frac{n\pi z'}{L}\right)d\phi'\,dz'}$$