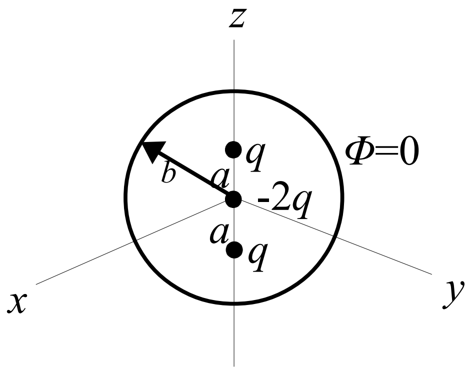
Jackson 3.7 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell

PROBLEM:

Three point charges (q, $-2q$, q) are located in a straight line with separation a and with the middle charge ($-2q$) at the origin of a grounded conducting spherical shell of radius b, as indicated in the sketch.



(a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates.

(b) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,

$$\Phi(r,\theta,\phi) \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1-\frac{r^5}{b^5}\right)P_2(\cos\theta)$$

SOLUTION:

(a) We already know the potential due to one point charge. We just add up the potential from each point charge:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-a(\hat{\mathbf{z}})|} + \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-a(-\hat{\mathbf{z}})|} - \frac{2q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}|}$$

Expand the first two terms using the Legendre polynomial expansions shown below in order to get a solution in spherical coordinates:

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \sum_{l=0}^{\infty}\frac{r_0^l}{r^{l+1}}P_l(\cos\theta) \quad \text{if } r > r_0$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\sum_{l=0}^{\infty}\frac{r^l}{r_0^{l+1}}P_l(\cos\theta) \quad \text{if } r<r_0$$

If $r>a$, $\Phi(\mathbf{r})=\frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta)+\frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{(-1)^l a^l}{r^{l+1}}P_l(\cos\theta)-\frac{2q}{4\pi\epsilon_0}\frac{1}{r}$

$$\Phi(\mathbf{r})=\frac{q}{4\pi\epsilon_0}\left[\frac{-2}{r}+\sum_{l=0}^{\infty}\left(1+(-1)^l\right)\frac{a^l}{r^{l+1}}P_l(\cos\theta)\right]$$

$$\boxed{\Phi(\mathbf{r})=\frac{2q}{4\pi\epsilon_0}\sum_{l=2,4,6\ldots}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta)} \quad (\text{if } r>a)$$

If $r<a$, $\Phi(\mathbf{r})=\frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{a^{l+1}}P_l(\cos\theta)+\frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}(-1)^l\frac{r^l}{a^{l+1}}P_l(\cos\theta)-\frac{2q}{4\pi\epsilon_0}\frac{1}{r}$

$$\boxed{\Phi(\mathbf{r})=\frac{2q}{4\pi\epsilon_0}\sum_{l=0,\text{even}}^{\infty}\frac{r^l}{a^{l+1}}\left[1-\frac{a}{r}\delta_{l,0}\right]P_l(\cos\theta)} \quad (\text{if } r<a)$$

Set $qa^2 = Q$ and keep it finite. As $a \to 0$, the only potential that matters is the $r>a$ case:

$$\Phi(\mathbf{r})=\left[\frac{2Q}{4\pi\epsilon_0}\sum_{l=0,2,4\ldots}^{\infty}\frac{a^{l-2}}{r^{l+1}}P_l(\cos\theta)\right]_{a\to 0}$$

As $a \to 0$, the higher terms in l get increasingly smaller, so that we only need to keep the first nonzero term:

$$\Phi(\mathbf{r})=\frac{2Q}{4\pi\epsilon_0}\left[\frac{1}{r^3}P_2(\cos\theta)+\ldots\right]$$

$$\boxed{\Phi(\mathbf{r})=\frac{Q}{4\pi\epsilon_0}\frac{1}{r^3}(3\cos^2\theta-1)}$$

The $l = 0$ term vanishes. This makes sense because the $l = 0$ term is the total charge (monopole) moment of the system and in this case the total charge is zero. The $l = 1$ term (the dipole moment term) drops out because of the symmetry of the charges. The first non-zero term is the quadrupole moment term ($l = 2$).

(b) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,

$$\Phi(r,\theta,\phi) \rightarrow \frac{Q}{2\pi\epsilon_0 r^3}\left(1-\frac{r^5}{b^5}\right)P_2(\cos\theta)$$

We don't have to explicitly place image charges. Rather we recognize that the image charges will create an additional potential to add to the potential of the original charges, and this image potential will be azimuthally symmetric and can thus be expanded in Legendre polynomials:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos\theta) + \frac{2q}{4\pi\epsilon_0}\sum_{l=2,4,6...}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta) \quad \text{if } r>a$$

Apply the boundary condition $\Phi(r=b)=0$

$$0 = \sum_l A_l b^l P_l(\cos\theta) + \frac{2q}{4\pi\epsilon_0}\sum_{l=2,4,6...}^{\infty}\frac{a^l}{b^{l+1}}P_l(\cos\theta)$$

$$0 = A_l b^l + \frac{2q}{4\pi\epsilon_0}\frac{a^l}{b^{l+1}} \quad \text{and} \quad A_l = 0 \quad \text{for } l = \text{odd and } l = 0$$

$$A_l = \frac{-2q}{4\pi\epsilon_0}\frac{a^l}{b^{2l+1}}$$

$$\boxed{\Phi(\mathbf{r}) = \sum_{l=2,\text{even}}^{\infty}\frac{2q}{4\pi\epsilon_0}\left(\frac{a}{b}\right)^l\left[\frac{b^l}{r^{l+1}}-\frac{r^l}{b^{l+1}}\right]P_l(\cos\theta)} \quad (\text{if } r>a)$$

In a similar manner we can solve the case of $r<a$

$$\boxed{\Phi(\mathbf{r}) = \frac{2q}{4\pi\epsilon_0}\sum_{l=0,\text{even}}^{\infty}\left(\frac{a}{b}\right)^l\left[\frac{-1}{b^{l+1}}+\frac{b^l}{a^{2l+1}}+\left(\frac{1}{b^{l+1}}-\frac{b^l}{a^{2l}}\frac{1}{r}\right)\delta_{l,0}\right]r^l P_l(\cos\theta)} \quad (\text{if } r<a)$$

Set $qa^2 = Q$ and keep it finite. As $a \rightarrow 0$, the only potential that matters is the $r>a$ case:

$$\Phi(\mathbf{r}) = \sum_{l=2,\text{even}}^{\infty}\frac{2Q}{4\pi\epsilon_0 a^2}\left(\frac{a}{b}\right)^l\left[\frac{b^l}{r^{l+1}}-\frac{r^l}{b^{l+1}}\right]P_l(\cos\theta)$$

As $a \rightarrow 0$, the higher terms in l get increasingly smaller, so that we only need to keep the first nonzero term:

$$\boxed{\Phi(\mathbf{r}) = \frac{Q}{2\pi\epsilon_0}\frac{1}{r^3}\left[1-\frac{r^5}{b^5}\right]P_2(\cos\theta)}$$