# Jackson 3.6 Homework Problem Solution

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**PROBLEM:**

Two point charges $q$ and $-q$ are located on the $z$ axis at $z = +a$ and $z = -a$, respectively.

(a) Find the electrostatic potential as an expansion in spherical harmonics and powers of $r$ for both $r > a$ and $r < a$.

(b) Keeping the product $qa = p/2$ constant, take the limit of $a \to 0$ and find the potential for $r \neq 0$. This is by definition a dipole along the $z$ axis and its potential.

(c) Suppose now that the dipole of part $b$ is surrounded by a *grounded* spherical shell of radius $b$ concentric with the origin. By linear superposition find the potential everywhere inside the shell.

**SOLUTION:**

(a) Using Coulomb's law, we can immediately write down the potential of two point charges:

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{r} - a\hat{\mathbf{k}}|} - \frac{1}{|\mathbf{r} + a\hat{\mathbf{k}}|}\right]$$

We can use the addition theorem to expand the $1/R$ factors:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \quad \text{where } r_< \text{ is smaller of } (r, r'), \text{ etc.}$$

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(0,0) Y_{lm}(\theta, \phi) - 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\pi, 0) Y_{lm}(\theta, \phi)\right]$$

where $r_<$ is smaller of $(r, a)$, etc.

$$\Phi = \frac{q}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi)\left[Y_{lm}^*(0,0) - Y_{lm}^*(\pi, 0)\right] \quad \text{where } r_< \text{ is smaller of } (r, a), \text{ etc.}$$

Written out explicitly, we have:

$$\boxed{\begin{aligned} \Phi &= \frac{q}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_l}{a^{l+1}} Y_{lm}(\theta, \phi)\left[Y_{lm}^*(0,0) - Y_{lm}^*(\pi, 0)\right] \quad \text{when } r < a \\ \Phi &= \frac{q}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{a^l}{r^{l+1}} Y_{lm}(\theta, \phi)\left[Y_{lm}^*(0,0) - Y_{lm}^*(\pi, 0)\right] \quad \text{when } r > a \end{aligned}}$$

The original question asked for the answer in terms of spherical harmonics, so that is what we have given. However, we can simplify the answer. Note that the problem is azimuthally symmetric, so that the solution must be azimuthally symmetric. This means that all terms in the series except the $m = 0$ term must vanish:

$$\Phi = \frac{q}{\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,0}(\theta,\phi)\left[Y_{l,0}^*(0,0) - Y_{l,0}^*(\pi,0)\right] \quad \text{where } r_< \text{ is smaller of } (r, a) \text{, etc.}$$

$$\Phi = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\left[P_l(1) - P_l(-1)\right] \quad \text{where } r_< \text{ is smaller of } (r, a) \text{, etc.}$$

$$\Phi = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\left[1 - (-1)^l\right] \quad \text{where } r_< \text{ is smaller of } (r, a) \text{, etc.}$$

$$\boxed{\Phi = \frac{2q}{4\pi\epsilon_0} \sum_{l=0,\,\text{odd}}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) \quad \text{where } r_< \text{ is smaller of } (r, a) \text{, etc.}}$$

(b) Keeping the product $qa = p/2$ constant, we take the limit of $a \rightarrow 0$ and find the potential for $r \neq 0$. We will always be in the region $r > a$:

$$\Phi = \frac{2q}{4\pi\epsilon_0} \sum_{l=0,\,\text{odd}}^{\infty} \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

$$\Phi = \frac{p}{4\pi\epsilon_0} \sum_{l=0,\,\text{odd}}^{\infty} \frac{a^{l-1}}{r^{l+1}} P_l(\cos\theta)$$

$$\Phi = \frac{p}{4\pi\epsilon_0}\left[\frac{1}{r^2} P_1(\cos\theta) + \frac{a^2}{r^4} P_3(\cos\theta) + \frac{a^4}{r^6} P_5(\cos\theta) + \ldots\right]$$

As $a \rightarrow 0$, this becomes:

$$\boxed{\Phi = \frac{p}{4\pi\epsilon_0} \frac{1}{r^2} \cos\theta}$$

This is the potential of a perfect dipole at the origin pointing along the $z$ axis.

(c) If the dipole of part $b$ is surrounded by a *grounded* spherical shell of radius $b$ concentric with the origin, the will be an extra potential due to the sphere and the total potential at the surface must be zero.

$$\Phi = \frac{p}{4\pi\epsilon_0} \frac{1}{r^2} \cos\theta$$

$$\Phi = \frac{p}{4\pi\epsilon_0}\frac{1}{r^2}\cos\theta + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Apply the boundary condition:

$$0 = \frac{p}{4\pi\epsilon_0}\frac{1}{b^2}\cos\theta + \sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta)$$

$$-\frac{p}{4\pi\epsilon_0}\frac{1}{b^2}\cos\theta = \sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta)$$

Due to orthogonality, only the $l = 1$ term will be nonzero.

$$A_1 = -\frac{p}{4\pi\epsilon_0}\frac{1}{b^3}$$

$$\boxed{\Phi = \frac{p\cos\theta}{4\pi\epsilon_0 b^2}\left[\left(\frac{b}{r}\right)^2 - \frac{r}{b}\right]}$$