Jackson 3.5 Homework Problem Solution

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PROBLEM:

A hollow sphere of inner radius *a* has the potential specified on its surface to be $\Phi = V(\theta, \phi)$. Prove the equivalence of the two forms of solution for the potential inside the sphere:

(a) $$\Phi(\mathbf{x}) = \frac{a(a^2 - r^2)}{4\pi} \int \frac{V(\theta', \phi')}{(r^2 + a^2 - 2ar\cos\gamma)^{3/2}} d\Omega'$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$.

(b) $$\Phi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \left(\frac{r}{a}\right)^l Y_{lm}(\theta, \phi) \quad \text{where} \quad A_{lm} = \int d\Omega' Y^*_{lm}(\theta', \phi') V(\theta', \phi')$$

SOLUTION:

The first solution is just the Green function method solution and the second is the Laplace equation solution. Let us solve both.

The spherical Green function was found using the method of images to be:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{\left|\frac{x'}{a}\mathbf{x} - \frac{a}{x'}\mathbf{x}'\right|}$$

In spherical coordinates this becomes:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{x^2 + x'^2 - 2xx'\cos\gamma}} - \frac{1}{\sqrt{\frac{x'^2}{a^2}x^2 + a^2 - 2xx'\cos\gamma}}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$

The normal derivative of the spherical Green function for the internal problem, evaluated at the sphere's surface, is found to be:

$$\left[\frac{dG}{dn'}\right]_{x'=a} = \frac{(x^2 - a^2)}{a(x^2 + a^2 - 2xa\cos\gamma)^{3/2}}$$

The general solution for the Green function method is:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\,G_D\,d^3\mathbf{x}' - \frac{1}{4\pi}\oint\left(\Phi\frac{dG_D}{dn'}\right)da'$$

There is no charge density, so this reduces down to:

$$\Phi(\mathbf{x}) = \frac{-1}{4\pi}\oint\left(\Phi\frac{dG_D}{dn'}\right)da'$$

$$\Phi(\mathbf{x}) = \frac{-1}{4\pi}\int\left(\Phi\frac{dG_D}{dn'}\right)a^2\,d\Omega'$$

$$\boxed{\Phi(\mathbf{x}) = \frac{a(a^2-r^2)}{4\pi}\int \frac{V(\theta',\phi')}{(r^2+a^2-2ra\cos\gamma)^{3/2}}\,d\Omega'}$$ where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

If we instead try to solve the Laplace equation, we start with the general solution (as derived previously) in terms of Spherical Harmonics Y_{lm}:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{l,m}r^l + B_{l,m}r^{-l-1}\right)Y_{lm}(\theta,\phi)$$

The region of interest includes the origin, so in order to have a valid solution there, $B_{l,m}$ must be zero

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m}r^l\,Y_{lm}(\theta,\phi)$$

Apply the boundary condition:

$$V(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m}a^l\,Y_{lm}(\theta,\phi)$$

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}(\theta,\phi)\,V(\theta,\phi)\sin\theta\,d\theta\,d\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m}a^l\int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}(\theta,\phi)\,Y_{lm}(\theta,\phi)\sin\theta\,d\theta\,d\phi$$

Use $\int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}(\theta,\phi)\,Y_{lm}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \delta_{l'l}\,\delta_{m'm}$

$$A_{l,m} = a^{-l}\int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}(\theta,\phi)\,V(\theta,\phi)\sin\theta\,d\theta\,d\phi$$

Finally:

$$\boxed{\Phi(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)}$$ where $$\boxed{A_{lm} = \int d\Omega'\,Y^*_{lm}(\theta',\phi')\,V(\theta',\phi')}$$

We have therefore *indirectly* shown that the two forms are equivalent because they are both valid solutions of the same problem.

We can also *directly* shown the two forms to be equivalent:
Start with the Green function method solution's form:

$$\Phi(\mathbf{x})=\frac{a(a^2-r^2)}{4\pi}\int\frac{V(\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}d\Omega'$$

We want to use the point potential expansion proved earlier:

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{r_0^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\quad\text{if}\quad r<r_0$$

If we take $\mathbf{r}_0$ to be a vector pointing to the location (a,θ',ϕ') then this becomes:

$$\frac{1}{\sqrt{r^2+a^2-2ar\cos\gamma}}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{a^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

Take the derivative of both sides with respect to *r* then multiply by *r*:

$$\frac{ar\cos\gamma-r^2}{(r^2+a^2-2ar\cos\gamma)^{3/2}}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l}{2l+1}\frac{r^l}{a^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

Take the derivative of the original expansion on both sides with respect to *a* and multiply by (-*a*):

$$\frac{-ar\cos\gamma+a^2}{(r^2+a^2-2ar\cos\gamma)^{3/2}}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l+1}{2l+1}\frac{r^l}{a^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

Add the two proceeding equations to each other:

$$\frac{a^2-r^2}{(r^2+a^2-2ar\cos\gamma)^{3/2}}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r^l}{a^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

Using this, expand part of the Green function method solution:

$$\Phi(\mathbf{x})=\int V(\theta',\phi')\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r^l}{a^l}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)d\Omega'$$

Finally:

$$\boxed{\Phi(\mathbf{x})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)}\quad\text{where}\quad\boxed{A_{lm}=\int d\Omega' Y^*_{lm}(\theta',\phi')V(\theta',\phi')}$$