PROBLEM:
The surface of a hollow conducting sphere of inner radius $a$ is divided into an even number of equal segments by a set of planes; their common line of intersection is the $z$ axis and they are distributed uniformly in the angle $\phi$. (The segments are like the skin on wedges of an apple, or the earth's surface between successive meridians of longitude.) The segments are kept at fixed potentials $\pm V$, alternately.

(a) Set up a series representation for the potential inside the sphere for the general case of $2n$ segments, and carry the calculation of the coefficients in the series far enough to determine exactly which coefficients are different from zero. For the non-vanishing terms, exhibit the coefficients as an integral over $\cos \theta$.

(b) For the special case of $n = 1$ (two hemispheres) determine explicitly the potential up to and including all terms with $l = 3$. By a coordinate transformation verify that this reduces to result (3.36) of Section 3.3.

SOLUTION:
There is no charge present, so we seek to solve Laplace's equation. In spherical coordinates this becomes:

$$\nabla^2 \Phi = 0 \rightarrow \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \Phi) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Using the method of separation of variables, and when the full azimuthal range needs a valid solution, the general solution is expressed in terms of the spherical harmonics $Y_{lm}$:

$$\Phi (r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A_{l,m} r^l + B_{l,m} r^{-l-1}) Y_{lm} (\theta, \phi)$$

In this problem, we require a valid solution at the origin, so that we must have $B_l = 0$ to keep those terms from blowing up. The solution now becomes:

$$\Phi (r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m} r^l Y_{lm} (\theta, \phi)$$

The boundary condition on the surface of the sphere is described mathematically as:
\[ \Phi(r=a) = V(\phi) \quad \text{where} \quad V(\phi) = \begin{cases} +V & \text{if } \frac{2i\pi}{n} < \phi < \frac{(2i+1)\pi}{n} \\ -V & \text{if } \frac{(2i+1)\pi}{n} < \phi < \frac{(2i+2)\pi}{n} \end{cases} \quad \text{where } i \text{ is any of } 0,1,...(n-1) \]

We apply this boundary condition:

\[ V(\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m} a^{l} Y_{l,m}(0,\phi) \]

Multiply both sides by \( Y_{l,m}^\star(0,\phi) \) and integrate over the surface of the sphere:

\[ \int_{0}^{2\pi} \int_{0}^{\pi} V(\phi) Y_{l,m}^\star(\theta,\phi) \sin \theta \, d\theta \, d\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m} a^{l} \int_{0}^{2\pi} \int_{0}^{\pi} Y_{l,m}^\star(\theta,\phi) Y_{l,m}(\theta,\phi) \sin \theta \, d\theta \, d\phi \]

Use the orthogonality of the spherical harmonics to pick one term from the double series:

\[ \int_{0}^{2\pi} \int_{0}^{\pi} V(\phi) Y_{l,m}^\star(\theta,\phi) \sin \theta \, d\theta \, d\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m} a^{l} \delta_{l,l} \delta_{m,m} \]

\[ \int_{0}^{2\pi} \int_{0}^{\pi} V(\phi) Y_{l,m}^\star(\theta,\phi) \sin \theta \, d\theta \, d\phi = A_{l,m} a^{l} \]

\[ A_{l,m} = a^{-l} \int_{0}^{2\pi} \int_{0}^{\pi} V(\phi) Y_{l,m}^\star(\theta,\phi) \sin \theta \, d\theta \, d\phi \]

Expand the definition of the spherical harmonics:

\[ A_{l,m} = a^{-l} \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} \int_{0}^{2\pi} V(\phi) e^{-im\phi} \, d\phi \int_{0}^{\pi} P_{l}^{m}(\cos \theta) \sin \theta \, d\theta \]

Break the integral over the azimuthal angle into a sum over \( 2n \) integral pieces and plug in the explicit value of the potential on the boundary:

\[ A_{l,m} = a^{-l} \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} \int_{0}^{\pi} P_{l}^{m}(\cos \theta) \sin \theta \, d\theta \left[ \sum_{j=0}^{2n-1} \int_{j\pi/n}^{(j+1)\pi/n} V(\phi) e^{-im\phi} \, d\phi \right] \]

\[ A_{l,m} = a^{-l} \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} \int_{0}^{\pi} P_{l}^{m}(\cos \theta) \sin \theta \, d\theta \left[ \sum_{j=0}^{2n-1} \int_{2j\pi/n}^{(2j+1)\pi/n} (V) e^{-im\phi} \, d\phi + \int_{2n\pi}^{(2n+1)\pi} (V) e^{-im\phi} \, d\phi \right] \]

\[ A_{l,m} = V a^{-l} \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} \int_{0}^{\pi} P_{l}^{m}(\cos \theta) \sin \theta \, d\theta \left[ \sum_{j=0}^{n-1} \int_{2j\pi/n}^{(2j+1)\pi/n} e^{-im\phi} \, d\phi - \int_{(2n+1)\pi/n}^{(2n+2)\pi/n} e^{-im\phi} \, d\phi \right] \]
Let us call the part in brackets $B_m$ and work it out separately:

$$B_m = \sum_{j=0}^{n-1} \left( \int_{\frac{2j\pi}{n}}^{\frac{(2j+1)\pi}{n}} e^{-im\Phi} d\Phi - \int_{\frac{(2j+2)\pi}{n}}^{\frac{(2j+3)\pi}{n}} e^{-im\Phi} d\Phi \right)$$

For $m = 0$ this reduces to:

$$B_0 = \sum_{j=0}^{n-1} \left( \int_{\frac{2j\pi}{n}}^{\frac{(2j+1)\pi}{n}} d\Phi - \int_{\frac{(2j+2)\pi}{n}}^{\frac{(2j+3)\pi}{n}} d\Phi \right)$$

$B_0 = 0$ and thus $A_{l,0} = 0$

For $m \neq 0$ we have:

$$B_m = \frac{i}{m} \sum_{j=0}^{n-1} e^{-im\frac{(2j+1)\pi}{n\lambda}} - e^{-im\frac{2j\pi}{n\lambda}} - e^{-im\frac{(2j+2)\pi}{n\lambda}} + e^{-im\frac{(2j+1)\pi}{n\lambda}}$$

$$B_m = \frac{i}{m} \sum_{j=0}^{n-1} e^{-im\frac{(2j+1)\pi}{n\lambda}} \left( 2e^{-im\frac{\pi}{n\lambda}} - 1 - e^{-im\frac{2\pi}{n\lambda}} \right)$$

$$B_m = -\frac{1}{m} \sum_{j=0}^{n-1} e^{-im\frac{(2j+1)\pi}{n\lambda}} \left( e^{-im\frac{\pi}{n\lambda}} - 1 \right)^2$$

Upon close inspection, if $\frac{m}{2n} = k$ where $k$ is some integer $k = ...,-2,-1,0,1,2,...$ then

$$B_m = -\frac{1}{m} \sum_{j=0}^{n-1} e^{-im\frac{(2j+1)\pi}{n\lambda}} \left( e^{-im\frac{\pi}{n\lambda}} - 1 \right)^2$$

$$B_m = \frac{1}{m} \sum_{j=0}^{n-1} \left( 1 - 1 \right)^2$$

$$B_m = 0 \quad \text{if} \quad \frac{m}{2n} = k$$

The only terms that do not vanish are the ones where $m/2n$ does not equate to an integer. Thus the terms that vanish are $m = 0, \pm 2n, \pm 4n, ...$

The final solution becomes:

$$\Phi(r, \theta, \Phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m} \left( \frac{r}{a} \right)^l Y_{lm}(\theta, \Phi)$$
\[
A_{l,m} = -V \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} e^{-i\pi l/m} \sum_{j=0}^{m-1} e^{-i(m(2j+1)/m)} \int_{0}^{\pi} P_{l}^{m} (\cos \theta) \sin \theta \, d\theta
\]

and \( A_{l,m} = 0 \) for \( m = 0, \pm 2n, \pm 4n \ldots \)

(b) For the special case of \( n = 1 \) (two hemispheres) determine explicitly the potential up to and including all terms with \( l = 3 \). By a coordinate transformation verify that this reduces to result (3.36) of Section 3.3.

\[
\Phi (r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l, \text{odd}}^{l} A_{l,m} \left( \frac{r}{a} \right)^{l} Y_{lm} (\theta, \phi)
\]

where \( A_{l,m} = -V \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} \int_{-1}^{1} P_{l}^{m} (x) \, dx \)

Expand this into terms up to \( l = 3 \):

\[
\Phi (r, \theta, \phi) = \sum_{m=-1, \text{odd}} A_{1,m} \left( \frac{r}{a} \right) Y_{1m} (0, \phi) + \sum_{m=-2, \text{odd}}^{2} A_{2,m} \left( \frac{r}{a} \right)^{2} Y_{2m} (0, \phi) + \sum_{m=-3, \text{odd}}^{3} A_{3,m} \left( \frac{r}{a} \right)^{3} Y_{3m} (0, \phi) + \ldots
\]

where the \( l = 0 \) term automatically drops out due to the fact that only coefficients for odd \( m \) are non-vanishing.

\[
\Phi (r, \theta, \phi) = \left( \frac{r}{a} \right) [A_{1,-1} Y_{1,-1} + A_{1,1} Y_{1,1}] + \left( \frac{r}{a} \right)^{2} [A_{2,-1} Y_{2,-1} + A_{2,1} Y_{2,1}] + \left( \frac{r}{a} \right)^{3} [A_{3,-3} Y_{3,-3} + A_{3,-1} Y_{3,-1} + A_{3,1} Y_{3,1} + A_{3,3} Y_{3,3}] + \ldots
\]

We must calculate the coefficients explicitly by expanding the definition of the Legendre polynomials and doing the integrals:

\[
A_{1,-1} = iV \sqrt{\frac{3\pi}{2}}, \quad A_{1,1} = iV \sqrt{\frac{3\pi}{2}}
\]

\[
A_{2,-1} = 0, \quad A_{2,1} = 0
\]

\[
A_{3,-3} = iV \sqrt{\frac{35\pi}{256}}, \quad A_{3,3} = iV \sqrt{\frac{35\pi}{256}}
\]

\[
A_{3,-1} = iV \sqrt{\frac{21\pi}{256}}, \quad A_{3,1} = iV \sqrt{\frac{21\pi}{256}}
\]
We plug these back in:

\[
\Phi (r, \theta , \phi ) = \left( \frac{r}{a} \right) i V \sqrt{\frac{3\pi}{2}} \left[ Y_{1,-1}(0, \phi ) + Y_{1,1}(0, \phi ) \right]
\]

\[
+ \left( \frac{r}{a} \right)^3 i V \left[ \frac{35\pi}{256} (Y_{3,-3} + Y_{3,3}(0, \phi )) + \frac{21\pi}{256} (Y_{3,-1} + Y_{3,1}) \right] + ...
\]

\[
\Phi (r, \theta , \phi ) = \frac{3}{2} V \left( \frac{r}{a} \right) \sin \theta \sin \phi + \left( \frac{r}{a} \right)^3 V \left[ \frac{35}{64} \sin^3 \theta \sin(3 \phi ) + \frac{21}{64} \sin \theta (5 \cos^2 \theta - 1) \sin(\phi ) \right] + ...
\]

For \( n = 1 \) only, this problem is actually azimuthally symmetric if we make a coordinate transformation:

\[ \cos \theta ' = \sin \theta \sin \phi \]

\[
\Phi (r, \theta , \phi ) = \frac{3}{2} V \left( \frac{r}{a} \right) \cos \theta ' - \frac{7}{8} \left( \frac{r}{a} \right)^3 V \left[ \frac{5}{2} \cos^3 \theta ' - \frac{3}{2} \cos \theta ' \right] + ...
\]

\[
\Phi (r, \theta , \phi ) = \frac{3}{2} V \left( \frac{r}{a} \right) P_1(\cos \theta ') - \frac{7}{8} \left( \frac{r}{a} \right)^3 V P_3(\cos \theta ') + ...
\]

This matches the solution found in the book using an azimuthal symmetry approach.