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Jackson 3.4 Homework Problem Solution

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PROBLEM:

The surface of a hollow conducting sphere of inner radius *a* is divided into an *even* number of equal segments by a set of planes; their common line of intersection is the *z* axis and they are distributed uniformly in the angle ϕ . (The segments are like the skin on wedges of an apple, or the earth's surface between successive meridians of longitude.) The segments are kept at fixed potentials $\pm V$, alternately.

(a) Set up a series representation for the potential inside the sphere for the general case of $2n$ segments, and carry the calculation of the coefficients in the series far enough to determine exactly which coefficients are different from zero. For the non-vanishing terms, exhibit the coefficients as an integral over $\cos\theta$.

(b) For the special case of $n = 1$ (two hemispheres) determine explicitly the potential up to and including all terms with $l = 3$. By a coordinate transformation verify that this reduces to result (3.36) of Section 3.3.

SOLUTION:

There is no charge present, so we seek to solve Laplace's equation. In spherical coordinates this becomes:

$$\nabla^2\Phi=0 \quad\rightarrow\quad \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

Using the method of separation of variables, and when the full azimuthal range needs a valid solution, the general solution is expressed in terms of the spherical harmonics Y_{lm}:

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(A_{l,m}r^l+B_{l,m}r^{-l-1})Y_{lm}(\theta,\phi)$$

In this problem, we require a valid solution at the origin, so that we must have $B_l = 0$ to keep those terms from blowing up. The solution now becomes:

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{l,m}r^lY_{lm}(\theta,\phi)$$

The boundary condition on the surface of the sphere is described mathematically as:

$$\Phi(r=a)=V(\phi) \quad \text{where} \quad V(\phi)=\begin{Bmatrix} +V \ \text{if}\ \frac{2i\pi}{n}<\phi<\frac{(2i+1)\pi}{n} \\ -V \ \text{if}\ \frac{(2i+1)\pi}{n}<\phi<\frac{(2i+2)\pi}{n} \end{Bmatrix} \quad \text{where } i \text{ is any of } 0,1,...(n\text{-}1)$$

We apply this boundary condition:

$$V(\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m} a^l Y_{lm}(\theta,\phi)$$

Multiply both sides by $Y^*_{l'm'}(\theta,\phi)$ and integrate over the surface of the sphere:

$$\int_0^{2\pi}\int_0^{\pi} V(\phi) Y^*_{l'm'}(\theta,\phi)\sin\theta\, d\theta\, d\phi=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m} a^l \int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

Use the orthogonality of the spherical harmonics to pick one term from the double series:

$$\int_0^{2\pi}\int_0^{\pi} V(\phi) Y^*_{l'm'}(\theta,\phi)\sin\theta\, d\theta\, d\phi=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m} a^l \delta_{l'l}\delta_{m'm}$$

$$\int_0^{2\pi}\int_0^{\pi} V(\phi) Y^*_{lm}(\theta,\phi)\sin\theta\, d\theta\, d\phi=A_{l,m} a^l$$

$$A_{l,m}=a^{-l}\int_0^{2\pi}\int_0^{\pi} V(\phi) Y^*_{lm}(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

Expand the definition of the spherical harmonics:

$$A_{l,m}=a^{-l}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\int_0^{2\pi} V(\phi) e^{-im\phi} d\phi \int_0^{\pi} P_l^m(\cos\theta)\sin\theta\, d\theta$$

Break the integral over the azimuthal angle into a sum over $2n$ integral pieces and plug in the explicit value of the potential on the boundary:

$$A_{l,m}=a^{-l}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\int_0^{\pi} P_l^m(\cos\theta)\sin\theta\, d\theta\left[\sum_{j=0}^{2n-1}\int_{j\pi/n}^{(j+1)\pi/n} V(\phi) e^{-im\phi} d\phi\right]$$

$$A_{l,m}=a^{-l}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\int_0^{\pi} P_l^m(\cos\theta)\sin\theta\, d\theta\left[\sum_{j=0}^{n-1}\left(\int_{2j\pi/n}^{(2j+1)\pi/n} (+V) e^{-im\phi} d\phi+\int_{(2j+1)\pi/n}^{(2j+2)\pi/n} (-V) e^{-im\phi} d\phi\right)\right]$$

$$A_{l,m}=V a^{-l}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\int_0^{\pi} P_l^m(\cos\theta)\sin\theta\, d\theta\left[\sum_{j=0}^{n-1}\left(\int_{2j\pi/n}^{(2j+1)\pi/n} e^{-im\phi} d\phi-\int_{(2j+1)\pi/n}^{(2j+2)\pi/n} e^{-im\phi} d\phi\right)\right]$$

Let us call the part in brackets B_m and work it out separately:

$$B_m=\sum_{j=0}^{n-1}\left(\int_{2j\pi/n}^{(2j+1)\pi/n} e^{-im\phi}\,d\phi-\int_{(2j+1)\pi/n}^{(2j+2)\pi/n} e^{-im\phi}\,d\phi\right)$$

For $m = 0$ this reduces to:

$$B_0=\sum_{j=0}^{n-1}\left(\int_{2j\pi/n}^{(2j+1)\pi/n} d\phi-\int_{(2j+1)\pi/n}^{(2j+2)\pi/n} d\phi\right)$$

$B_0=0$ and thus $A_{l,0}=0$

For $m \neq 0$ we have:

$$B_m=\frac{i}{m}\sum_{j=0}^{n-1}\left(e^{-im(2j+1)\pi/n}-e^{-im2j\pi/n}-e^{-im(2j+2)\pi/n}+e^{-im(2j+1)\pi/n}\right)$$

$$B_m=\frac{i}{m}\sum_{j=0}^{n-1}e^{-im(2j)\pi/n}\left(2e^{-im\pi/n}-1-e^{-im(2)\pi/n}\right)$$

$$B_m=\frac{-i}{m}\sum_{j=0}^{n-1}e^{-im(2j)\pi/n}\left(e^{-im\pi/n}-1\right)^2$$

Upon close inspection, if $\frac{m}{2n}=k$ where k is some integer k = ...,-2, -1, 0, 1, 2,... then

$$B_m=\frac{-i}{m}\sum_{j=0}^{n-1}e^{-im(2j)\pi/n}\left(e^{-i2\pi k}-1\right)^2$$

$$B_m=\frac{i}{m}\sum_{j=0}^{n-1}(1-1)^2$$

$$B_m=0 \quad \text{if} \quad \frac{m}{2n}=k$$

The only terms that do not vanish are the ones where $m/2n$ does not equate to an integer. Thus the terms that vanish are $m=0,\pm 2n,\pm 4n,\ldots$

The final solution becomes:

$$\boxed{\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{l,m}\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)}$$

where $\boxed{A_{l,m}=-V\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\frac{i}{m}\left(e^{-im\pi/n}-1\right)^2\sum_{j=0}^{n-1}e^{-im(2j)\pi/n}\int_0^{\pi}P_l^m(\cos\theta)\sin\theta\,d\theta}$

and $A_{l,m}=0$ for $m=0,\pm 2n,\pm 4n...$

(b) For the special case of $n = 1$ (two hemispheres) determine explicitly the potential up to and including all terms with $l = 3$. By a coordinate transformation verify that this reduces to result (3.36) of Section 3.3.

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l,\text{odd}}^{l}A_{l,m}\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)$$

where $A_{l,m}=-V\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\frac{i}{m}4\int_{-1}^{1}P_l^m(x)\,dx$

Expand this into terms up to $l = 3$:

$$\Phi(r,\theta,\phi)=\sum_{m=-1,\text{odd}}^{1}A_{1,m}\left(\frac{r}{a}\right)Y_{1m}(\theta,\phi)+\sum_{m=-2,\text{odd}}^{2}A_{2,m}\left(\frac{r}{a}\right)^2 Y_{2m}(\theta,\phi)+\sum_{m=-3,\text{odd}}^{3}A_{3,m}\left(\frac{r}{a}\right)^3 Y_{3m}(\theta,\phi)+...$$

where the $l = 0$ term automatically drops out due to the fact that only coefficients for odd m are non-vanishing.

$$\Phi(r,\theta,\phi)=\left(\frac{r}{a}\right)\left[A_{1,-1}Y_{1,-1}+A_{1,1}Y_{1,1}\right]+\left(\frac{r}{a}\right)^2\left[A_{2,-1}Y_{2-1}+A_{2,1}Y_{21}\right]$$
$$+\left(\frac{r}{a}\right)^3\left[A_{3,-3}Y_{3,-3}+A_{3,-1}Y_{3,-1}+A_{3,1}Y_{3,1}+A_{3,3}Y_{3,3}\right]+...$$

We must calculate the coefficients explicitly by expanding the definition of the Legendre polynomials and doing the integrals:

$$A_{1,-1}=iV\sqrt{\frac{3\pi}{2}}\quad,\quad A_{1,1}=iV\sqrt{\frac{3\pi}{2}}$$

$$A_{2,-1}=0\quad,\quad A_{2,1}=0$$

$$A_{3,-3}=iV\sqrt{\frac{35\pi}{256}}\quad,\quad A_{3,3}=iV\sqrt{\frac{35\pi}{256}}$$

$$A_{3,-1}=iV\sqrt{\frac{21\pi}{256}}\quad,\quad A_{3,1}=iV\sqrt{\frac{21\pi}{256}}$$

We plug these back in:

$$\Phi(r,\theta,\phi)=\left(\frac{r}{a}\right)iV\sqrt{\frac{3\pi}{2}}\left[Y_{1,-1}(\theta,\phi)+Y_{1,1}(\theta,\phi)\right]$$
$$+\left(\frac{r}{a}\right)^3 iV\left[\sqrt{\frac{35\pi}{256}}(Y_{3,-3}+Y_{3,3}(\theta,\phi))+\sqrt{\frac{21\pi}{256}}(Y_{3,-1}+Y_{3,1})\right]+\ldots$$

$$\Phi(r,\theta,\phi)=\frac{3}{2}V\left(\frac{r}{a}\right)\sin\theta\sin\phi+\left(\frac{r}{a}\right)^3 V\left[\frac{35}{64}\sin^3\theta\sin(3\phi)+\frac{21}{64}\sin\theta(5\cos^2\theta-1)\sin(\phi)\right]+\ldots$$

For $n = 1$ only, this problem is actually azimuthally symmetric if we make a coordinate transformation:

$$\cos\theta'=\sin\theta\sin\phi$$

$$\Phi(r,\theta,\phi)=\frac{3}{2}V\left(\frac{r}{a}\right)\cos\theta'-\frac{7}{8}\left(\frac{r}{a}\right)^3 V\left[\frac{5}{2}\cos^3\theta'-\frac{3}{2}\cos\theta'\right]+\ldots$$

$$\boxed{\Phi(r,\theta,\phi)=\frac{3}{2}V\left(\frac{r}{a}\right)P_1(\cos\theta')-\frac{7}{8}\left(\frac{r}{a}\right)^3 V P_3(\cos\theta')+\ldots}$$

This matches the solution found in the book using an azimuthal symmetry approach.