Jackson 3.2 Homework Problem Solution

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PROBLEM:

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

where, for $l = 0$, $P_{l-1}(\cos\alpha) = -1$. What is the potential outside?

(b) Find the magnitude and the direction of the electric field at the origin.

(c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

SOLUTION:

Because of the spherical geometry of the problem and the absence of charge in the region where we want to find the potential, we choose to solve the Laplace equation in spherical coordinates. Using separation of variables, the general solution for azimuthal symmetry becomes:

$$\Phi(r,\theta,\phi) = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

The problem does not specify a boundary condition, only a charge distribution. We can convert this to a boundary condition using the Gaussian pillbox method:

$$\left[(\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}})\cdot\mathbf{r} = \frac{1}{\epsilon_0}\sigma\right]_{n=n_0}$$

$$\sigma = \epsilon_0\left[-\frac{\partial\Phi_{\text{out}}}{\partial r} + \frac{\partial\Phi_{\text{in}}}{\partial r}\right]_{r=R} \quad \text{where} \quad \sigma = \frac{Q}{4\pi R^2} \text{ if } \theta > \alpha \quad \text{and} \quad \sigma = 0 \text{ if } \theta < \alpha$$

It becomes apparent that we must solve the potential both on the inside and the outside at the same times and apply multiple boundary conditions.
For the potential inside the sphere, there must be a valid solution at the origin. This forces $B_l = 0$.

$$\Phi_{\text{in}} = \sum_l A_l r^l P_l(\cos\theta)$$

For the potential outside the sphere, there must be a finite solution at infinity, this forces $A_l = 0$ for $l > 0$

$$\Phi_{\text{out}} = A_{0,\text{out}} + \sum_l B_l r^{-l-1} P_l(\cos\theta)$$

The A_0 term in the outer potential is obviously the value of the potential at infinity. The potential of a localized charge distribution at infinity is the potential of a point charge with a charge magnitude of the total charge of the distribution.

$$\Phi_{\text{out}}(r \to \infty) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{tot}}}{r}$$

$$\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{tot}}}{r} = \left[A_{0,\text{out}} + \sum_l B_l r^{-l-1} P_l(\cos\theta)\right]_{r\to\infty}$$

$$\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{tot}}}{r} = A_{0,\text{out}} + B_0\frac{1}{r}$$

This must be true for all r, so that $A_{0,\text{out}} = 0$ and $B_0 = \frac{Q_{\text{tot}}}{4\pi\epsilon_0}$

We can calculate the total charge by integrating over the charge density:

$$Q_{\text{tot}} = \int_0^{2\pi}\int_0^{\pi} \sigma R^2 \sin(\theta)\, d\theta\, d\phi$$

$$Q_{\text{tot}} = \frac{Q}{4\pi R^2}\int_0^{2\pi}\int_{\alpha}^{\pi} R^2 \sin(\theta)\, d\theta\, d\phi$$

$$Q_{\text{tot}} = \frac{Q}{2}(\cos(\alpha)+1)$$

$$B_0 = \frac{Q}{8\pi\epsilon_0}(\cos(\alpha)+1)$$

$$\Phi_{\text{out}} = (\cos(\alpha)+1)\frac{Q}{8\pi\epsilon_0 r} + \sum_{l=1}^{\infty} B_l r^{-l-1} P_l(\cos\theta)$$

The potential must be continuous so that $\Phi_{\text{in}}(r=R) = \Phi_{\text{out}}(r=R)$. This leads to

$$A_0 + \sum_{l=1}^{\infty} A_l R^l P_l(\cos\theta) = (\cos(\alpha)+1)\frac{Q}{8\pi\epsilon_0 R} + \sum_{l=1}^{\infty} B_l R^{-l-1} P_l(\cos\theta)$$

This must hold for all values of θ, so that every coefficient in the series must equate:

$$A_0=(\cos(\alpha)+1)\frac{Q}{8\pi\epsilon_0 R} \quad \text{and} \quad B_l=A_l R^{2l+1}$$

The potential inside and outside now becomes:

$$\Phi_{\text{in}}=(\cos(\alpha)+1)\frac{Q}{8\pi\epsilon_0 R}+\sum_{l=1}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_{\text{out}}=(\cos(\alpha)+1)\frac{Q}{8\pi\epsilon_0 r}+\sum_{l=1}^{\infty} A_l R^{2l+1} r^{-l-1} P_l(\cos\theta)$$

Now apply the last boundary condition:

$$\sigma=\epsilon_0\left[-\frac{\partial\Phi_{\text{out}}}{\partial r}+\frac{\partial\Phi_{\text{in}}}{\partial r}\right]_{r=R}$$

$$\sigma=\epsilon_0\left[-[-(\cos(\alpha)+1)\frac{Q}{8\pi\epsilon_0 r^2}+\sum_{l=1}^{\infty} A_l R^{2l+1}(-l-1)r^{-l-2}P_l(\cos\theta)]+[\sum_{l=1}^{\infty} A_l l r^{l-1} P_l(\cos\theta)]\right]_{r=R}$$

$$\sigma=(\cos(\alpha)+1)\frac{Q}{8\pi R^2}+\epsilon_0\sum_{l=1}^{\infty} A_l R^{l-1}(2l+1)P_l(\cos\theta)$$

Multiply both sides by $P_{l'}(\cos\theta)\sin\theta$ where $l' > 0$ and integrate

$$\int_0^{\pi}\sigma P_{l'}(\cos\theta)\sin\theta\, d\theta=\epsilon_0\sum_{l=1}^{\infty} A_l R^{l-1}(2l+1)\int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$\int_0^{\pi}\sigma P_l(\cos\theta)\sin\theta\, d\theta=2\epsilon_0 A_l R^{l-1}$$

$$A_l=\frac{1}{2\epsilon_0}R^{-l+1}\int_0^{\pi}\sigma P_l(\cos\theta)\sin\theta\, d\theta$$

$$A_l=\frac{1}{2\epsilon_0}R^{-l+1}\frac{Q}{4\pi R^2}\int_{\alpha}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta$$

$$A_l=\frac{Q}{8\pi\epsilon_0}R^{-l-1}\frac{1}{2l+1}\int_{-1}^{\cos(\alpha)}\frac{d}{dx}[P_{l+1}(x)-P_{l-1}(x)]dx$$

$$A_l=\frac{Q}{8\pi\epsilon_0}R^{-l-1}\frac{1}{2l+1}[P_{l+1}(x)-P_{l-1}(x)]_{-1}^{\cos(\alpha)}$$

$$A_l=\frac{Q}{8\pi\epsilon_0}R^{-l-1}\frac{1}{2l+1}[P_{l+1}(\cos(\alpha))-P_{l-1}(\cos(\alpha))-P_{l+1}(-1)+P_{l-1}(-1)]$$

$$\boxed{A_l = \frac{Q}{8\pi\epsilon_0} R^{-l-1} \frac{1}{2l+1} [P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]}$$

The constants are now all found and the final solution becomes:

$$\Phi_{\text{in}} = (\cos(\alpha)+1)\frac{Q}{8\pi\epsilon_0 R} + \sum_{l=1}^{\infty} \frac{Q}{8\pi\epsilon_0} \frac{1}{2l+1} [P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))] \frac{r^l}{R^{l+1}} P_l(\cos\theta)$$

The $l = 0$ term can be combined with the other terms if we make the definition $P_{-1}(\cos\alpha) = -1$

$$\boxed{\Phi_{\text{in}} = \frac{Q}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} [P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))] \frac{r^l}{R^{l+1}} P_l(\cos\theta)}$$

$$\boxed{\Phi_{\text{out}} = \frac{Q}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} [P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))] \frac{R^l}{r^{l+1}} P_l(\cos\theta)}$$

(b) Find the magnitude and the direction of the electric field at the origin.

$$\mathbf{E}(r=0) = [-\nabla\Phi_{\text{in}}]_{r=0}$$

$$\mathbf{E}(r=0) = -\left[\hat{\mathbf{r}}\frac{\partial\Phi_{\text{in}}}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Phi_{\text{in}}}{\partial\theta}\right]_{r=0}$$

$$\mathbf{E}(r=0) = -[\hat{\mathbf{r}}\frac{\partial}{\partial r}\frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$
$$+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]\frac{r^l}{R^{l+1}}P_l(\cos\theta)]_{r=0}$$

One must be careful to realize that the $l = 0$ term does not depend on r or θ, so that its derivative with respect to r or θ is zero.

$$\mathbf{E}(r=0) = -[\hat{\mathbf{r}}\sum_{l=1}^{\infty}\frac{Q}{8\pi\epsilon_0}\frac{l}{2l+1}[P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]\frac{r^{l-1}}{R^{l+1}}P_l(\cos\theta)$$
$$+\hat{\boldsymbol{\theta}}\sum_{l=1}^{\infty}\frac{Q}{8\pi\epsilon_0}\frac{1}{2l+1}[P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]\frac{r^{l-1}}{R^{l+1}}\frac{(l\cos\theta P_l(\cos\theta) - l P_{l-1}(\cos\theta))}{\sin\theta}]_{r=0}$$

The $l = 2$ terms and higher all equate to zero at the origin. Only the $l = 1$ terms are independent or r and do not zero out.

$$\mathbf{E}(r=0) = -\frac{Q}{24\pi\epsilon_0 R^2}[P_2(\cos(\alpha)) - P_0(\cos(\alpha))][\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta]$$

$$\boxed{\mathbf{E}(r=0)=\frac{Q\sin^2(\alpha)}{16\pi\epsilon_0 R^2}\hat{\mathbf{z}}}$$

(c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

As the spherical cap becomes very small, $\alpha \rightarrow 0$ and $\cos(\alpha) \rightarrow (1- \alpha^2/2)$.

$$P_l(\cos(\alpha))\approx P_l(1-\frac{1}{2}\alpha^2)$$

Use the Taylor series expansion: $P_l(x)=P_l(1)+P'_l(1)(x-1)$

$$P_l(\cos(\alpha))\approx 1-\frac{1}{2}\alpha^2 P'_l(1)$$

We can use this to expand the term we will need:

$$P_{l+1}(\cos(\alpha))-P_{l-1}(\cos(\alpha))\approx 1-\frac{1}{2}\alpha^2 P'_{l+1}(1)-1+\frac{1}{2}\alpha^2 P'_{l-1}(1)$$
$$P_{l+1}(\cos(\alpha))-P_{l-1}(\cos(\alpha))\approx -\frac{1}{2}\alpha^2(P'_{l+1}(1)-P'_{l-1}(1))$$

$$P_{l+1}(\cos(\alpha))-P_{l-1}(\cos(\alpha))\approx -\frac{1}{2}\alpha^2(2l+1)$$

We must be careful and handle the $l = 0$ case separately:

$$P_1(\cos(\alpha))-P_{-1}(\cos(\alpha))=\cos(\alpha)+1$$

$$P_1(\cos(\alpha))-P_{-1}(\cos(\alpha))\approx 2-\frac{1}{2}\alpha^2$$

Plug these expansions into the solutions found in (part a)

$$\Phi_{\text{in}}=\frac{Q}{8\pi\epsilon_0}[2-\frac{1}{2}\alpha^2]\frac{1}{R}+\frac{Q}{8\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{1}{2l+1}[-\frac{1}{2}\alpha^2(2l+1)]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

$$\Phi_{\text{in}}=\frac{Q}{4\pi\epsilon_0 R}-\frac{Q\alpha^2}{16\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

The sum is just the Green function expansion of the potential due to a unit point charge on the z axis, so that:

$$\boxed{\Phi_{\text{in}} = \frac{Q}{4\pi\epsilon_0 R} - \frac{Q\alpha^2}{16\pi\epsilon_0}\frac{1}{|\mathbf{r} - R\hat{\mathbf{z}}|}}$$

This physically corresponds to the potential due to a complete charged sphere plus the potential due to an oppositely charged point particle at the point where the positive z axis crosses the sphere.

The potential outside the sphere is found in a similar way and gives similar results:

$$\Phi_{\text{out}} = \frac{Q}{8\pi\epsilon_0}[2 - \frac{1}{2}\alpha^2]\frac{1}{r} + \frac{Q}{8\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{1}{2l+1}[-\frac{1}{2}\alpha^2(2l+1)]\frac{R^l}{r^{l+1}}P_l(\cos\theta)$$

$$\Phi_{\text{out}} = \frac{Q}{4\pi\epsilon_0 r} - \frac{Q\alpha^2}{16\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{R^l}{r^{l+1}}P_l(\cos\theta)$$

$$\boxed{\Phi_{\text{out}} = \frac{Q}{4\pi\epsilon_0 r} - \frac{Q\alpha^2}{16\pi\epsilon_0}\frac{1}{|\mathbf{r} - R\hat{\mathbf{z}}|}}$$

The electric field at the origin becomes, when using $\sin(\alpha)\approx\alpha$

$$\boxed{\mathbf{E}(r=0) = \frac{Q\alpha^2}{16\pi\epsilon_0 R^2}\hat{\mathbf{z}}}$$

The other extreme is when $\alpha \to \pi$. Define the angle of the small cap of charge as α_2 so that $\alpha = \pi - \alpha_2$ and the problem now to apply $\alpha_2 \to 0$. We have $\cos(\alpha) = \cos(\pi - \alpha_2) = -\cos(\alpha_2) \to -(1 - \alpha_2^2/2)$. With the problem set up this way, the approach is almost identical to that above.

$$P_l(\cos(\alpha)) = (-1)^l P_l(1 - \frac{1}{2}\alpha_2^2)$$

$$P_l(\cos(\alpha)) \approx (-1)^l\left(1 - \frac{1}{2}\alpha_2^2 P'_l(1)\right)$$

$$P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha)) \approx (-1)^l\frac{1}{2}\alpha_2^2(2l+1)$$

$$\Phi_{\text{in}} = \frac{Q\alpha_2^2}{16\pi\epsilon_0}\sum_{l=0}^{\infty}(-1)^l\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

$$\Phi_{\text{in}} = \frac{Q\alpha_2^2}{16\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(-\cos\theta)$$

$$\boxed{\Phi_{\text{in}} = \frac{Q\alpha_2^2}{16\pi\epsilon_0}\frac{1}{|\mathbf{r} - (-R\hat{\mathbf{z}})|}}$$

$$\boxed{\Phi_{\text{out}} = \frac{Q\alpha_2^2}{16\pi\epsilon_0}\frac{1}{|\mathbf{r}-(-R\hat{\mathbf{z}})|}}$$

These solutions correspond physically to a point charge at the point where the negative z axis crosses the sphere.

The electric field at the origin becomes:

$$\boxed{\mathbf{E}(r=0) = \frac{Q\alpha_2^2}{16\pi\epsilon_0 R^2}\hat{\mathbf{z}}}$$

Note that the electric field at the origin due to a point charge at $z = -R$ is the same as that due to an oppositely charged point charge at $z = R$.