Jackson 3.1 Homework Problem Solution

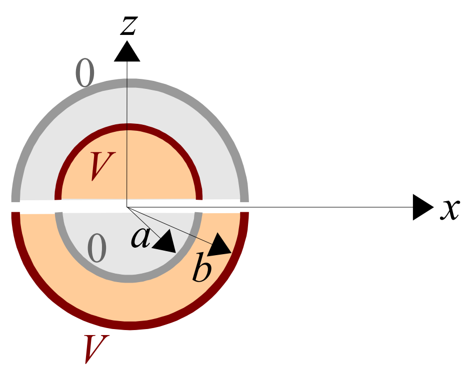
Dr. Christopher S. Baird
University of Massachusetts Lowell

PROBLEM:

Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential. Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting case $b \to \infty$, and $a \to 0$.

SOLUTION:

The geometry of the problem can be sketched as a cross-section of the spheres:



Because of the spherical geometry of the problem and the absence of charge, we choose to solve the Laplace equation in spherical coordinates:

$$\nabla^2 \Phi = 0 \quad \rightarrow \quad \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Using the method of separation of variables leads to the general solution:

$$\Phi(r,\theta,\phi) = \sum_l (A_l r^l + B_l r^{-l-1})(A_{m=0} + B_{m=0}\phi) P_l^{m=0}(\cos\theta)$$
$$+ \sum_{m\neq 0, l} (A_l r^l + B_l r^{-l-1})(A_m e^{im\phi} + B_m e^{-im\phi}) P_l^m(\cos\theta)$$

The boundaries in this problem are all azimuthally symmetric, so that the solution for the electric potential will not be a function of ϕ . The only way to make the general solution independent of ϕ is to set $m = 0$ and $B_{m=0} = 0$.

$$\Phi(r,\theta,\phi) = \sum_l (A_l r^l + B_l r^{-l-1}) P_l^{m=0}(\cos\theta)$$

In other problems where we must have a valid solution at the origin, B_l must be zero to keep the solution from blowing up at the origin. But in this problem, we do not seek a valid solution at the origin, so we cannot use this restriction.

Apply the boundary condition: $\Phi(r=a)=V_1$ where $V_1=V$ if $\theta<\pi/2$ and $V_1=0$ if $\theta>\pi/2$

$$V_1=\sum_l (A_l a^l+B_l a^{-l-1}) P_l(\cos\theta)$$

Multiply both sides by $P_{l'}(\cos\theta)\sin\theta$ and integrate

$$\int_0^\pi V_1 P_{l'}(\cos\theta)\sin\theta\, d\theta=\sum_l (A_l a^l+B_l a^{-l-1})\int_0^\pi P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\, d\theta$$

Use the orthogonality condition of Legendre polynomials: $\int_0^\pi P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\, d\theta=\frac{2}{2l+1}\delta_{l'l}$

$$\int_0^\pi V_1 P_{l'}(\cos\theta)\sin\theta\, d\theta=\sum_l (A_l a^l+B_l a^{-l-1})\frac{2}{2l+1}\delta_{l'l}$$

The Kronecker delta makes every term in the sum collapse to zero except when $l=l'$

$$\int_0^\pi V_1 P_l(\cos\theta)\sin\theta\, d\theta=(A_l a^l+B_l a^{-l-1})\frac{2}{2l+1}$$

Flip the equation and plug in the explicit form of V_1

$$A_l a^l+B_l a^{-l-1}=\frac{2l+1}{2}V\int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta$$

Make a change of variables: $x=\cos\theta$, $dx=-\sin\theta\, d\theta$

$$\boxed{A_l a^l+B_l a^{-l-1}=\frac{2l+1}{2}V\int_0^1 P_l(x)\,dx}$$

Apply the boundary condition: $\Phi(r=b)=V_2$ where $V_2=0$ if $\theta<\pi/2$ and $V_2=V$ if $\theta>\pi/2$

$$V_2=\sum_l (A_l b^l+B_l b^{-l-1}) P_l(\cos\theta)$$

Multiply both sides by $P_{l'}(\cos\theta)\sin\theta$ and integrate

$$\int_0^\pi V_2 P_{l'}(\cos\theta)\sin\theta\, d\theta=\sum_l (A_l b^l+B_l b^{-l-1})\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

Use the orthogonality condition of Legendre polynomials: $\int_0^{\pi} P_{l'}(\cos\theta)\, P_l(\cos\theta)\sin\theta\, d\,\theta = \frac{2}{2l+1}\delta_{l'l}$

$$A_l b^l + B_l b^{-l-1} = \frac{2l+1}{2}\int_0^{\pi} V_2 P_l(\cos\theta)\sin\theta\, d\,\theta$$

Plug in the explicit form of V_2

$$A_l b^l + B_l b^{-l-1} = \frac{2l+1}{2} V \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\,\theta$$

Make a change of variables: $x = \cos\theta$, $dx = -\sin\theta\, d\,\theta$

$$A_l b^l + B_l b^{-l-1} = \frac{2l+1}{2} V \int_{-1}^{0} P_l(x)\, dx$$

Make a change of variables $x \rightarrow -x$ and use the identity: $P_l(-x) = (-1)^l P_l(x)$

$$\boxed{A_l b^l + B_l b^{-l-1} = \frac{2l+1}{2} V (-1)^l \int_0^1 P_l(x)\, dx}$$

The two equations in boxes form a system of equations that we can solve for A_l and B_l.

$$\boxed{A_l = \frac{(-a^{l+1} + (-1)^l b^{l+1})}{(-a^{2l+1} + b^{2l+1})}\left(\frac{2l+1}{2}\right) V \int_0^1 P_l(x) dx}$$

$$\boxed{B_l = \frac{a^{l+1} b^{2l+1} - (-1)^l a^{2l+1} b^{l+1}}{-a^{2l+1} + b^{2l+1}}\left(\frac{2l+1}{2}\right) V \int_0^1 P_l(x) dx}$$

The final solution is then:

$$\Phi(r,\theta,\phi) = \frac{V}{2}\sum_l (2l+1)\int_0^1 P_l(x)\, dx \frac{[(a^{l+1} - (-1)^l b^{l+1}) r^l - a\, b (a^l b^{2l} - (-1)^l a^{2l} b^l) r^{-l-1}]}{a^{2l+1} - b^{2l+1}} P_l(\cos\theta)$$

We can do the integral, but we must be careful to do the $l = 0$ and the $l > 0$ cases separately:

$l = 0$: $\quad \int_0^1 P_0(x)\, dx = \int_0^1 dx = 1$

$l > 0$: $\quad \int_0^1 P_l(x) dx = \frac{1}{2l+1}\int_0^1 \frac{d}{dx}[P_{l+1}(x) - P_{l-1}(x)]\, dx$

$$\int_0^1 P_l(x)dx = \frac{1}{2l+1}[P_{l+1}(x) - P_{l-1}(x)]_0^1$$

$$\int_0^1 P_l(x)dx = \frac{1}{2l+1}[P_{l+1}(1) - P_{l-1}(1) - P_{l+1}(0) + P_{l-1}(0)]$$

$$\int_0^1 P_l(x)dx = \frac{1}{2l+1}[P_{l-1}(0) - P_{l+1}(0)]$$

When the $l = 0$ term is taken out of the sum and the integral solutions plugged in, the final solution becomes:

$$\Phi(r,\theta,\phi) = \frac{V}{2} + \frac{V}{2}\sum_{l=1}^{\infty}[P_{l-1}(0) - P_{l+1}(0)]\frac{[(a^{l+1} - (-1)^l b^{l+1})r^l - ab(a^l b^{2l} - (-1)^l a^{2l} b^l)r^{-l-1}]}{a^{2l+1} - b^{2l+1}} P_l(\cos\theta)$$

It should be noted that $[P_{l-1}(0) - P_{l+1}(0)] = 0$ when l is even, so that all the even terms drop out. Because l is only odd, then $(-1)^l = -1$

$$\boxed{\Phi(r,\theta,\phi) = \frac{V}{2} + \frac{V}{2}\sum_{l=1,\text{odd}}^{\infty}[P_{l-1}(0) - P_{l+1}(0)]\frac{[(a^{l+1} + b^{l+1})r^l - a^{l+1} b^{l+1}(b^l + a^l)r^{-l-1}]}{a^{2l+1} - b^{2l+1}} P_l(\cos\theta)}$$

This can be expressed explicitly as an expanded sum (up to term $l = 4$ is shown):

$$\Phi(r,\theta,\phi) = \frac{V}{2}[1 + \left(\frac{3}{2}\right)\frac{[(a^2 + b^2)r - a^2 b^2(b+a)r^{-2}]}{a^3 - b^3}\cos\theta \\ + \left(\frac{-7}{16}\right)\frac{[(a^4 + b^4)r^3 - a^4 b^4(b^3 + a^3)r^{-4}]}{a^7 - b^7}(5\cos^3\theta - 3\cos\theta) + \ldots]$$

When we make a statement about the behavior of the solution as $b \to \infty$, we are really making a statement about b being very large compared to something. So we first get the solution in a form where b is always the denominator of a ratio:

$$\Phi(r,\theta,\phi) = \frac{V}{2} + \frac{V}{2}\sum_{l=1,\text{odd}}^{\infty}[P_{l-1}(0) - P_{l+1}(0)]\frac{[((a/b)^{l+1} + 1)(r/b)^l - (1 + (a/b)^l)(a/r)^{l+1}]}{(a/b)^{2l+1} - 1} P_l(\cos\theta)$$

Now we can make the statement that as $b \to \infty$, $(a/b) \to 0$, and $(r/b) \to 0$ so that:

$$\boxed{\Phi(r,\theta,\phi) = \frac{V}{2} + \frac{V}{2}\sum_{l=1,\text{odd}}^{\infty}[P_{l-1}(0) - P_{l+1}(0)]\left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)}$$

This can be expressed explicitly as an expanded sum (up to term $l = 4$ is shown):

$$\Phi(r,\theta,\phi) = \frac{V}{2}\left[1 + \left(\frac{3}{2}\right)\left(\frac{a}{r}\right)^2\cos\theta + \left(\frac{-7}{16}\right)\left(\frac{a}{r}\right)^4(5\cos^3\theta - 3\cos\theta) + \ldots\right]$$

We can check this by solving the problem outright. With azimuthal symmetry, the full angular range included, and the condition that the solution does not blow up at infinity, the general solution becomes:

$$\Phi(r,\theta,\phi)=A_0+\sum_l B_l r^{-l-1} P_l(\cos\theta)$$

Apply the boundary condition (derived by physical arguments): $\Phi(r\to\infty)=V/2$

$$V/2=A_0$$

Apply the boundary condition: $\Phi(r=a)=V_1$ where $V_1=V$ if $\theta<\pi/2$ and $V_1=0$ if $\theta>\pi/2$

$$V_1=\frac{V}{2}+\sum_l B_l a^{-l-1} P_l(\cos\theta)$$

Multiply both sides by $P_{l'}(\cos\theta)\sin\theta\, d\,\theta$ where $l' > 0$ and integrate.

$$\int_0^\pi V_1 P_l(\cos\theta)\sin\theta\, d\,\theta=B_l a^{-l-1}\frac{2}{2l+1}$$

$$B_l=\frac{2l+1}{2}a^{l+1}\int_0^\pi V_1 P_l(\cos\theta)\sin\theta\, d\,\theta$$

$$B_l=\frac{2l+1}{2}V\, a^{l+1}\int_0^1 P_l(x)dx$$

$$B_l=\frac{V}{2}a^{l+1}[P_{l-1}(0)-P_{l+1}(0)] \quad \text{where } l \text{ is odd}$$

$$\boxed{\Phi(r,\theta,\phi)=\frac{V}{2}+\frac{V}{2}\sum_{l=1,\text{odd}}[P_{l-1}(0)-P_{l+1}(0)]\left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)}$$

This matches the solution found above.

The behavior of the original solution as $a \to 0$ is:

$$\Phi(r,\theta,\phi)=\frac{V}{2}+\frac{V}{2}\sum_{l=1,\text{odd}}^{\infty}[P_{l-1}(0)-P_{l+1}(0)]\frac{[(a^{l+1}+b^{l+1})r^l-a^{l+1}b^{l+1}(b^l+a^l)r^{-l-1}]}{a^{2l+1}-b^{2l+1}}P_l(\cos\theta)$$

$$\boxed{\Phi(r,\theta,\phi)=\frac{V}{2}-\frac{V}{2}\sum_{l=1,\text{odd}}^{\infty}[P_{l-1}(0)-P_{l+1}(0)]\left(\frac{r}{b}\right)^{l} P_l(\cos\theta)}$$

This can be expressed explicitly as an expanded sum (up to term $l = 4$ is shown):

$$\Phi(r,\theta,\phi)=\frac{V}{2}\left[1-\left(\frac{3}{2}\right)\frac{r}{b}\cos\theta-\left(\frac{-7}{16}\right)\left(\frac{r}{b}\right)^3(5\cos^3\theta-3\cos\theta)+...\right]$$

We can check this by solving the problem outright. With azimuthal symmetry, the full angular range included, and the condition that the solution does not blow up at the origin, the general solution becomes:

$$\Phi(r,\theta,\phi)=\sum_l A_l r^l P_l(\cos\theta)$$

Apply the boundary condition: $\Phi(r=b)=V_2$ where $V_2=0$ if $\theta<\pi/2$ and $V_2=V$ if $\theta>\pi/2$

$$V_2=\sum_l A_l b^l P_l(\cos\theta)$$

Multiply both sides by $P_{l'}(\cos\theta)\sin\theta\, d\,\theta$ and integrate.

$$\int_0^\pi V_2 P_{l'}(\cos\theta)\sin\theta\, d\,\theta=\sum_l A_l b^l\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\,\theta$$

$$\int_0^\pi V_2 P_l(\cos\theta)\sin\theta\, d\,\theta=A_l b^l\frac{2}{2l+1}$$

$$A_l=\frac{2l+1}{2}b^{-l}\int_0^\pi V_2 P_l(\cos\theta)\sin\theta\, d\,\theta$$

$$A_l=\frac{2l+1}{2}V b^{-l}\int_{-1}^0 P_l(x)\,dx$$

$$A_l=\frac{2l+1}{2}V b^{-l}(-1)^l\int_0^1 P_l(x)\,dx$$

$$A_l=\frac{-V}{2}b^{-l}[P_{l-1}(0)-P_{l+1}(0)] \quad \text{where } l \text{ is odd, and } A_0=\frac{V}{2}$$

$$\boxed{\Phi(r,\theta,\phi)=\frac{V}{2}-\frac{V}{2}\sum_{l=1,\text{odd}}[P_{l-1}(0)-P_{l+1}(0)]\left(\frac{r}{b}\right)^l P_l(\cos\theta)}$$

This matches the solution found above.