Jackson 3.15 Homework Problem Solution

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PROBLEM:
Consider the following "spherical cow" model of a battery connected to an external circuit. A sphere of radius a and conductivity σ is embedded in a uniform medium of conductivity σ'. Inside the sphere there is a uniform (chemical) force in the z direction acting on the charge carriers; its strength as an effective electric field entering Ohm's law is F. In the steady state, electric fields exist inside and outside the sphere and surface charge resides on its surface.

(a) Find the electric field (in addition to F) and current density everywhere in space. Determine the surface-charge density and show that the electric dipole moment of the sphere is $p = 4\pi\varepsilon_0\sigma a^3 F/(\sigma + 2\sigma')$.

(b) Show that the total current flowing out through the upper hemisphere of the sphere is

$$I = \frac{2\sigma\sigma'}{\sigma+2\sigma'}\pi a^2 F$$

Calculate the total power dissipation outside the sphere. Using the lumped circuit relations, $P = I^2R_e = IV_e$, find the effective external resistance R_e and voltage V_e.

(c) Find the power dissipated within the sphere and deduce the effective internal resistance R_i and V_i.

(d) Define the total voltage through the relation $V_t = (R_e + R_i)I$ and show that $V_t = 4aF/3$, as well as $V_e + V_i = V_t$. Show that IV_t is the power supplied by the "chemical" force.

SOLUTION:
(a) There is a total electric field $\mathbf{E}_{\text{ext}}$ external to the sphere and there is a total internal field $\mathbf{E}_{\text{int}}$. The boundary surface is a sphere and the entire problem has azimuthal symmetry, so the potential inside and outside must have the form:

$$\Phi_{\text{int}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{and} \quad \Phi_{\text{ext}} = \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta)$$

We link the two by applying boundary conditions at $r = a$:

$$(\mathbf{E}_{\text{ext}} - \mathbf{E}_{\text{int}})\cdot\hat{\mathbf{r}} = \sigma_s/\epsilon_0 \quad \text{and} \quad \hat{\mathbf{r}}\times\mathbf{E}_{\text{ext}} = \hat{\mathbf{r}}\times\mathbf{E}_{\text{int}}$$

where σ_s is the surface charge density, not to be confused with the conductivity σ. Let us look at the first boundary condition:

$$(-\nabla\Phi_{\text{ext}} + \nabla\Phi_{\text{int}})\cdot\hat{\mathbf{r}} = \sigma_s/\epsilon_0$$

$$-\frac{\partial \Phi_{\text{ext}}}{\partial r}+\frac{\partial \Phi_{\text{int}}}{\partial r}=\sigma_s/\epsilon_0$$

$$-\sum_{l=0}^{\infty} B_l(-l-1)a^{-l-2}P_l(\cos\theta)+\sum_{l=0}^{\infty} A_l l\, a^{l-1} P_l(\cos\theta)=\sigma_s/\epsilon_0$$

$$-\sum_{l=0}^{\infty} B_l(-l-1)a^{-l-2}P_l(\cos\theta)+\sum_{l=0}^{\infty} A_l l\, a^{l-1} P_l(\cos\theta)=1/\epsilon_0\sum_{l=0}^{\infty} C_l P_l(\cos\theta)$$

where $\sigma_s=\sum_{l=0}^{\infty} C_l P_l(\cos\theta)$

$$\boxed{B_l(l+1)a^{-l-2}+A_l l\, a^{l-1}=C_l/\epsilon_0}$$

The other boundary condition is:

$$\hat{\mathbf{r}}\times\nabla\Phi_{\text{ext}}=\hat{\mathbf{r}}\times\nabla\Phi_{\text{int}}$$

$$\frac{\partial \Phi_{\text{ext}}}{\partial \theta}=\frac{\partial \Phi_{\text{int}}}{\partial \theta}$$

$$\boxed{B_l=A_l a^{2l+1}}$$

Using both equations above in boxes, we find:

$$\boxed{A_l=\frac{C_l}{\epsilon_0 a^{l-1}(2l+1)}} \quad \boxed{B_l=\frac{C_l a^{l+2}}{\epsilon_0(2l+1)}}$$

Our solution now becomes:

$$\Phi_{\text{int}}=\frac{a}{\epsilon_0}\sum_{l=0}^{\infty}\frac{C_l}{2l+1}\left(\frac{r}{a}\right)^l P_l(\cos\theta) \quad \text{and} \quad \Phi_{\text{ext}}=\frac{a}{\epsilon_0}\sum_{l=0}^{\infty}\frac{C_l}{2l+1}\left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)$$

where $C_l=\frac{2l+1}{2}\int_0^{\pi}\sigma_s(\theta)P_l(\cos\theta)\sin\theta\, d\theta$

This is the general solution. Given any charge density on the surface of a sphere, we can use this solution to find the fields.

In this problem, the chemical force inside the sphere acts as an effective electric field in addition to the actual electric field and is given as uniform and pointing in the *z* direction:

$$\mathbf{F}(\mathbf{x})=F\,\hat{\mathbf{z}}$$

Because the actual internal electric field is in response to the chemical force (without the chemical force, there would be no electric fields anywhere), and because of the symmetry, we can assume the internal electric field is also uniform and in the z direction.

$$\mathbf{E}_{\text{int}} = E_{\text{int}}\hat{\mathbf{z}}$$

$$\Phi_{\text{int}} = -E_{\text{int}} r\cos\theta$$

Set this equal to the general expression:

$$-E_{\text{int}} r\cos\theta = \frac{a}{\epsilon_0}\sum_{l=0}^{\infty}\frac{C_l}{2l+1}\left(\frac{r}{a}\right)^l P_l(\cos\theta)$$

Due to orthogonality, only the the $l = 1$ term can be non-zero.

$\boxed{C_1 = -3\epsilon_0 E_{\text{int}}}$ and $C_l = 0$ for $l \neq 1$

Our solution is now:

$$\Phi_{\text{int}} = -E_{\text{int}} r\cos\theta \quad , \quad \Phi_{\text{ext}} = -E_{\text{int}}\frac{a^3}{r^2}\cos\theta \quad , \quad \sigma_s = -3\epsilon_0 E_{\text{int}}\cos\theta$$

or

$$\mathbf{E}_{\text{int}} = E_{\text{int}}\hat{\mathbf{z}} \quad , \quad \mathbf{E}_{\text{ext}} = (\hat{\mathbf{z}} - 3\hat{\mathbf{r}}\cos\theta) E_{\text{int}}\frac{a^3}{r^3} \quad , \quad \sigma_s = -3\epsilon_0 E_{\text{int}}\cos\theta$$

Ohm's law states $\mathbf{J} = \sigma\mathbf{E}$. The internal current is a result of the chemical force and the internal electric field:

$$\mathbf{J}_{\text{int}} = \sigma(\mathbf{E}_{\text{int}} + \mathbf{F}) \quad \text{and} \quad \mathbf{J}_{\text{ext}} = \sigma'\mathbf{E}_{\text{ext}}$$

$$\mathbf{J}_{\text{int}} = \sigma(E_{\text{int}} + F)\hat{\mathbf{z}} \quad \text{and} \quad \mathbf{J}_{\text{ext}} = \sigma'(\hat{\mathbf{z}} - 3\hat{\mathbf{r}}\cos\theta) E_{\text{int}}\frac{a^3}{r^3}$$

The component of the current normal to the sphere's surface must be continuous to ensure we are in a steady-state configuration:

$$\left[\mathbf{J}_{\text{int}}\cdot\hat{\mathbf{r}} = \mathbf{J}_{\text{ext}}\cdot\hat{\mathbf{r}}\right]_{r=a}$$

$$\sigma(E_{\text{int}} + F) = -\sigma' 2 E_{\text{int}}$$

$$E_{\text{int}} = -\frac{\sigma}{\sigma + 2\sigma'} F$$

Our solutions now become:

$$\boxed{\begin{aligned}&\mathbf{E}_{\text{int}}=-\frac{\sigma}{\sigma+2\sigma'}F\,\hat{\mathbf{z}}\quad,\quad \mathbf{E}_{\text{ext}}=\frac{\sigma}{\sigma+2\sigma'}F\left(3\cos\theta\,\hat{\mathbf{r}}-\hat{\mathbf{z}}\right)\frac{a^3}{r^3}\quad,\quad \sigma_s=3\epsilon_0\frac{\sigma}{\sigma+2\sigma'}F\cos\theta\\ &\mathbf{J}_{\text{int}}=\frac{2\sigma\sigma'}{\sigma+2\sigma'}F\,\hat{\mathbf{z}}\quad,\quad \mathbf{J}_{\text{ext}}=\left(3\cos\theta\,\hat{\mathbf{r}}-\hat{\mathbf{z}}\right)\frac{\sigma'\sigma}{\sigma+2\sigma'}F\frac{a^3}{r^3}\end{aligned}}$$

Note that the external field has the same pattern as that due to a point dipole. Comparing the field in this problem to the field of a dipole:

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left[3\cos\theta\,\hat{\mathbf{r}}-\hat{\mathbf{z}}\right]$$

we see that the battery acts as an effective dipole with dipole moment:

$$\boxed{p=4\pi\epsilon_0 F a^3\frac{\sigma}{\sigma+2\sigma'}}$$

(b) The total current flowing out through the upper hemisphere is:

$$I=\int_0^{\pi/2}\int_0^{2\pi}\mathbf{J}_{\text{int}}\cdot\hat{\mathbf{r}}\,a^2\sin\theta\,d\phi\,d\theta$$

$$I=a^2F\frac{2\sigma\sigma'}{\sigma+2\sigma'}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$$\boxed{I=\frac{2\sigma\sigma'}{\sigma+2\sigma'}\pi a^2F}$$

Because the conductivity in both regions is finite, some of the energy is lost to Joule heating of the materials. The power dissipated by loosing electromagnetic energy to mechanical energy when creating currents is:

$$P=\int\mathbf{J}\cdot\mathbf{E}\,dV$$

Using Ohm's law, this becomes:

$$P_{\text{int}}=\frac{1}{\sigma}\int J_{\text{int}}^2\,dV\quad\text{and}\quad P_{\text{ext}}=\frac{1}{\sigma'}\int J_{\text{ext}}^2\,dV$$

$$P_{\text{int}}=\frac{16\sigma\sigma'^2}{3(\sigma+2\sigma')^2}F^2\pi a^3\quad\text{and}\quad P_{\text{ext}}=\frac{\sigma'^2\sigma^2}{(\sigma+2\sigma')^2}F^2a^6\frac{1}{\sigma'}\int_0^{2\pi}\int_0^{\pi}\int_a^{\infty}\left(3\cos^2\theta+1\right)\frac{1}{r^4}\sin\theta\,dr\,d\theta\,d\phi$$

$$\boxed{P_{\text{int}}=\frac{16\sigma\sigma'^2}{3(\sigma+2\sigma')^2}F^2\pi a^3}$$ and $$\boxed{P_{\text{ext}}=\frac{8\sigma'\sigma^2}{3(\sigma+2\sigma')^2}F^2\pi a^3}$$

The total power dissipated is therefore:

$$P=P_{\text{int}}+P_{\text{ext}}$$

$$\boxed{P=\frac{8\sigma\sigma'}{3(\sigma+2\sigma')}F^2\pi a^3}$$

The effective external resistance is therefore:

$$R_{\text{ext}}=\frac{P_{\text{ext}}}{I^2}$$

$$\boxed{R_{\text{ext}}=\frac{2}{3\sigma'\pi a}}$$

The effective external voltage is:

$$V_{\text{ext}}=\frac{P_{\text{ext}}}{I}$$

$$\boxed{V_{\text{ext}}=\frac{4\sigma}{3(\sigma+2\sigma')}F a}$$

(c) The power dissipated inside the sphere was already found in the previous step to be:

$$\boxed{P_{\text{int}}=\frac{16\sigma\sigma'^2}{3(\sigma+2\sigma')^2}F^2\pi a^3}$$

$$R_{\text{int}}=\frac{P_{\text{int}}}{I^2}$$

$$\boxed{R_{\text{int}}=\frac{4}{3\sigma\pi a}}$$

$$V_{\text{int}}=\frac{P_{\text{int}}}{I}$$

$$\boxed{V_{\text{int}}=\frac{8\sigma'}{3(\sigma+2\sigma')}F a}$$

(d) The total voltage is: $V_t = (R_e + R_i)I$

$$V_t = \left(\frac{2}{3\sigma' \pi a} + \frac{4}{3\sigma \pi a} \right) \frac{2\sigma\sigma'}{\sigma + 2\sigma'} \pi a^2 F$$

$$\boxed{V_t = \frac{4}{3} a F}$$

We can also calculate it as the sum of the external and internal voltages:

$$V_t = V_{\text{ext}} + V_{\text{int}}$$

$$V_t = \frac{4\sigma}{3(\sigma + 2\sigma')} F a + \frac{8\sigma'}{3(\sigma + 2\sigma')} F a$$

$$\boxed{V_t = \frac{4}{3} F a}$$

The total power supplied by the chemical force is: $P_t = IV_t$

$$\boxed{P_t = \frac{8\sigma\sigma'}{3(\sigma + 2\sigma')} \pi a^3 F^2}$$

This matches the total power dissipated by the materials.