Jackson 3.14 Homework Problem Solution

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PROBLEM:

A line charge of length $2d$ with a total charge Q has a linear charge density varying as $(d^2 - z^2)$, where z is the distance from the midpoint. A grounded, conducting, spherical shell of inner radius $b > d$ is centered at the midpoint of the line charge.

(a) Find the potential everywhere inside the spherical shell as an expansion in Legendre polynomials.

(b) Calculate the surface-charge density induced on the shell.

(c) Discuss your answers to part a and b in the limit that $d << b$.

SOLUTION:

This problem includes both a charge distribution and a boundary condition. We must use the Green function method to include both. The Green function method solution for Dirichlet boundary conditions is:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')G_D d^3\mathbf{x}' - \frac{1}{4\pi}\oint\left(\Phi\frac{d\,G_D}{d\,n'}\right)da'$$

In this particular case, the boundary condition states that the potential is zero on the spherical surface. This causes the entire surface integral in the Green function solution to equate to zero. The solution is then:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')G_D d^3\mathbf{x}'$$

The charge density that satisfies the description above is:

$$\rho(r,\theta,\phi) = \frac{3Q}{8\pi d^3}\frac{(d^2-r^2)}{r^2}[\delta(\cos\theta-1)+\delta(\cos\theta+1)] \quad \text{if } r<d \text{ and } \quad \rho(r,\theta,\phi)=0 \quad r>d$$

The r^2 in the denominator was required to convert the linear charge density into spherical coordinates. The constants out front were found by integrating over all space and setting the result equal to the total charge Q. The solution now becomes:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\rho(\mathbf{x}')G_D r'^2\sin\theta' dr' d\theta' d\phi'$$

$$\Phi(\mathbf{x})=\frac{3Q}{16\pi\epsilon_0 d^3}\left[\int_0^d (d^2-r'^2)G_D(\theta'=0)\,dr'+\int_0^d (d^2-r'^2)\,G_D(\theta'=\pi)\,dr'\right]$$

The Green function for this problem is found by placing a unit point charge inside a conducting sphere, using the image method to simulate the effects of the conducting sphere, writing out the potential due to the point charge and the image charge, and expanding the solution in spherical harmonics. The Green function then results:

$$G_D=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(r r')^l}{b^{2l+1}}\right]Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

This problem has azimuthal symmetry, so it is obvious that all terms in the sum must be zero except the $m=0$ terms. After expanding the definition of the spherical harmonics and keeping only the $m=0$ terms, the Green function simplifies to:

$$G_D=\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(r r')^l}{b^{2l+1}}\right]P_l(\cos\theta')P_l(\cos\theta)$$

The integral needs the Green function at only certain angles, so we evaluate at those angles now:

$$G_D(\theta'=0)=\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(r r')^l}{b^{2l+1}}\right]P_l(\cos\theta)\quad\text{and}\quad G_D(\theta'=\pi)=\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(r r')^l}{b^{2l+1}}\right](-1)^l P_l(\cos\theta)$$

We can now evaluate the final solution:

$$\Phi(\mathbf{x})=\frac{3Q}{8\pi\epsilon_0 d^3}\sum_{l=0,\text{even}}^{\infty}P_l(\cos\theta)\int_0^d dr'(d^2-r'^2)\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(r r')^l}{b^{2l+1}}\right]$$

If $r>d$

$$\Phi(\mathbf{x})=\frac{3Q}{8\pi\epsilon_0 d^3}\sum_{l=0,\text{even}}^{\infty}P_l(\cos\theta)\int_0^d dr'(d^2-r'^2)\left[\frac{r'^l}{r^{l+1}}-\frac{(r r')^l}{b^{2l+1}}\right]$$

$$\boxed{\Phi(\mathbf{x})=\frac{3Q}{4\pi\epsilon_0}\sum_{l=0,\text{even}}^{\infty}P_l(\cos\theta)\frac{d^l}{(l+1)(l+3)b^{l+1}}\left[\left(\frac{b}{r}\right)^{l+1}-\left(\frac{r}{b}\right)^l\right]}$$

If $r<d$ the integral must be split into two parts, the $r>r'$ part and the $r<r'$ part:

$$\Phi(\mathbf{x})=\frac{3Q}{8\pi\epsilon_0 d^3}\sum_{l=0,\text{even}}^{\infty}P_l(\cos\theta)[\int_0^r dr'(d^2-r'^2)\left[\frac{r'^l}{r^{l+1}}-\frac{(r r')^l}{b^{2l+1}}\right]+\int_r^d dr'(d^2-r'^2)\left[\frac{r^l}{r'^{l+1}}-\frac{(r r')^l}{b^{2l+1}}\right]]$$

$$\boxed{\Phi(\mathbf{x})=\frac{3Q}{8\pi\epsilon_0 d^3}\sum_{l=0,\text{even}}^{\infty} P_l(\cos\theta)\left[d^2\left(\frac{2l+1}{l(l+1)}\right)+r^2\left(\frac{2l+1}{(2-l)(3+l)}\right)+r^l\left(\frac{1}{d^{l-2}}\left(\frac{2}{l(l-2)}\right)-\left(\frac{2}{(l+3)(l+1)}\right)\frac{d^{l+3}}{b^{2l+1}}\right)\right]}$$

(b) Calculate the surface-charge density induced on the shell.

$$\sigma=\left[-\epsilon_0\frac{\partial\Phi}{\partial n}\right]_{\text{on S}}$$

We want the surface-charge density induced on the inner side of the shell, so that it's normal points inwards, in the opposite direction of the radial dimension: $n = -r$

$$\sigma=\left[\epsilon_0\frac{\partial\Phi}{\partial r}\right]_{r=b}$$

The potential is needed at the surface $r = b$, which is greater than d, so we use the $r > d$ solution:

$$\sigma=\left[\epsilon_0\frac{3Q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\sum_{l=0,\text{even}}^{\infty} P_l(\cos\theta)\frac{d^l}{(l+1)(l+3)b^{l+1}}\left[\left(\frac{b}{r}\right)^{l+1}-\left(\frac{r}{b}\right)^l\right]\right]_{r=b}$$

$$\boxed{\sigma=-\frac{3Q}{4\pi b^2}\sum_{l=0,\text{even}}^{\infty} P_l(\cos\theta)\frac{(2l+1)}{(l+1)(l+3)}\left(\frac{d}{b}\right)^l}$$

(c) Discuss your answers to part a and b in the limit that $d << b$.

When $d << b$, the potential of greatest importance becomes the case where $r > d$ which was found as:

$$\Phi(\mathbf{x})=\frac{3Q}{4\pi\epsilon_0 b}\sum_{l=0,\text{even}}^{\infty} P_l(\cos\theta)\frac{1}{(l+1)(l+3)}\left(\frac{d}{b}\right)^l\left[\left(\frac{b}{r}\right)^{l+1}-\left(\frac{r}{b}\right)^l\right]$$

The fact that $d << b$ means that $(d/b) << 1$ so that the term $(d/b)^l$ becomes zero for all terms except $l = 0$

$$\boxed{\Phi(\mathbf{x})=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r}-\frac{1}{b}\right]}$$

This is just the potential due to a point charge Q at the center of a conducting sphere of radius b.

The exact same reasoning applies to the surface-charge density so that only the $l = 0$ term remains:

$$\boxed{\sigma=-\frac{Q}{4\pi b^2}}$$

This is just a total charge $-Q$ spread out uniformly over a sphere of radius b.