Jackson 3.12 Homework Problem Solution

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PROBLEM:

An infinite, thin, plane sheet of conducting material has a circular hole of radius a cut in it. A thin, flat disc of the same material and slightly smaller radius lies in the plane, filling the hole, but separated from the sheet by a very narrow insulating ring. The disc is maintained at a fixed potential V, while the infinite sheet is kept at zero potential.

(a) Using appropriate cylindrical coordinates, find an integral expression involving Bessel functions for the potential at any point above the plane.

(b) Show that the potential a perpendicular distance z above the *center* of the disc is

$$\Phi_0(z)=V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)$$

(c) Show that the potential a perpendicular distance z above the *edge* of the disc is

$$\Phi_a(z)=\frac{V}{2}\left(1-\frac{k\,z}{\pi a}K(k)\right)$$

where $k = 2a/(z^2 + 4a^2)^{1/2}$, and $K(k)$ is the complete elliptic integral of the first kind.

SOLUTION:

(a) The general solution to the Laplace equation in cylindrical coordinates is:

$$\begin{aligned}\Phi(\rho,\phi,z)=&(A_{0,0}+B_{0,0}\ln\rho)(C_{0,0}+D_{0,0}\phi)(F_{0,0}+G_{0,0}z)\\&+\sum_{\nu\neq 0}(A_{\nu,0}\rho^{\nu}+B_{\nu,0}\rho^{-\nu})(C_{\nu,0}e^{i\nu\phi}+D_{\nu,0}e^{-i\nu\phi})(F_{\nu,0}+G_{\nu,0}z)\\&+\sum_{k\neq 0}(A_{0,k}J_0(k\rho)+B_{0,k}N_0(k\rho))(C_{0,k}+D_{0,k}\phi)(F_{0,k}e^{kz}+G_{0,k}e^{-kz})\\&+\sum_{\nu\neq 0}\sum_{k\neq 0}(A_{\nu,k}J_\nu(k\rho)+B_{\nu,k}N_\nu(k\rho))(C_{\nu,k}e^{i\nu\phi}+D_{\nu,k}e^{-i\nu\phi})(F_{\nu,k}e^{kz}+G_{\nu,k}e^{-kz})\end{aligned}$$

The problem has azimuthal symmetry, immediately dictating that $v = 0$ and the function be single-valued in the azimuthal angle, leading to:

$$\Phi(\rho,\phi,z)=(A_0+B_0\ln\rho)(F_0+G_0z)+\sum_{k\neq 0}(A_kJ_0(k\rho)+B_kN_0(k\rho))(F_ke^{kz}+G_ke^{-kz})$$

The solution must be finite as z approaches infinity, telling us that $F_k = 0$ and $G_0 =0$.
Also, the solution must be finite along the z axis ($\rho = 0$), telling us that $B_k = 0$ and $B_0 = 0$. Our solution is now:

$$\Phi(\rho,\phi,z)=\sum_k A_k J_0(k\rho)e^{-kz}$$

Typically at this point, we would apply a boundary condition at a finite radius, but there is not one for this problem. With no constraint on the radius, *k* becomes a continuous spectrum and not a discrete set.

$$\Phi(\rho,\phi,z)=\int_0^\infty A(k)J_0(k\rho)e^{-kz}dk$$

Apply the last boundary condition:

$$\Phi(z=0)=V(\rho)$$

$$V(\rho)=\int_0^\infty A(k)J_0(k\rho)dk$$

We can take advantage of the orthogonality statement to solve this:

$$\int_0^\infty x J_0(kx)J_0(k'x)dx=\frac{1}{k}\delta(k'-k) \quad \text{(Jackson 3.108)}$$

$$A(k)=k\int_0^\infty V(\rho)\rho J_0(k\rho)d\rho$$

Insert the actual boundary condition in this problem:

$$A(k)=Vk\int_0^a J_0(k\rho)\rho d\rho$$

$$A(k)=VaJ_1(ka)$$

The final solution is:

$$\boxed{\Phi(\rho,\phi,z)=Va\int_0^\infty J_1(ka)J_0(k\rho)e^{-kz}dk}$$

(b) The potential a perpendicular distance *z* above the *center* of the disc is:

$$\Phi(\rho=0)=Va\int_0^\infty J_1(ka)J_0(0)e^{-kz}dk$$

$$\Phi(\rho=0)=Va\int_0^\infty J_1(ka)e^{-kz}dk$$

$$\Phi(\rho=0)=V\int_0^\infty J_1(x)e^{-xz/a}dx$$

Use $J_1(x)=\frac{1}{2\pi i}\int_0^{2\pi} e^{i(x\cos\theta+\theta)}\,d\theta$

$$\Phi(\rho=0)=\frac{1}{2\pi i}V\int_0^{2\pi} e^{i\theta}\int_0^{\infty} e^{(i\cos\theta-z/a)x}\,dx\,d\theta$$

$$\Phi(\rho=0)=\frac{1}{2\pi}V\int_0^{2\pi} e^{i\theta}\frac{1}{(\cos\theta+i\,z/a)}\,d\theta$$

$$\Phi(\rho=0)=\frac{1}{2\pi}V\,[\int_0^{2\pi}\frac{\cos^2\theta+\sin\theta\,z/a}{(\cos^2\theta+z^2/a^2)}\,d\theta+i\int_0^{2\pi}\frac{-\cos\theta\,z/a+\sin\theta\cos\theta}{(\cos^2\theta+z^2/a^2)}\,d\theta]$$

Several pieces go away due to symmetry.

$$\Phi(\rho=0)=\frac{1}{2\pi}V\int_0^{2\pi}\frac{\cos^2\theta}{(\cos^2\theta+z^2/a^2)}\,d\theta$$

$$\Phi(\rho=0)=V\left[1-\frac{2}{\pi}\frac{z^2}{a^2}\int_0^{\pi/2}\frac{1}{\cos^2\theta+z^2/a^2}\,d\theta\right]$$

Substitute: $u=\frac{1}{\sqrt{1+a^2/z^2}}\tan\theta$, $\cos^2\theta=\frac{1}{1+(1+a^2/z^2)u^2}$, $d\theta=\frac{\sqrt{1+a^2/z^2}}{1+(1+a^2/z^2)u^2}\,du$

$$\Phi(\rho=0)=V\left[1-\frac{2}{\pi}\frac{z}{\sqrt{z^2+a^2}}\int_0^{\infty}\frac{1}{1+u^2}\,du\right]$$

$$\Phi(\rho=0)=V\left[1-\frac{2}{\pi}\frac{z}{\sqrt{z^2+a^2}}[\tan^{-1}(\infty)-\tan^{-1}(0)]\right]$$

$$\boxed{\Phi(\rho=0)=V\left[1-\frac{z}{\sqrt{a^2+z^2}}\right]}$$

(c) We can now find the potential a perpendicular distance *z* above the *edge* of the disc.

$$\Phi(\rho=a)=V\,a\int_0^{\infty} J_1(ka)J_0(k\,a)e^{-kz}\,dk$$

$$\Phi(\rho=a)=V\int_0^{\infty} J_1(x)J_0(x)e^{-x\,z/a}\,dx$$

Use $J_1(x)=\frac{1}{2\pi i}\int_0^{2\pi} e^{i(x\cos\theta+\theta)}\,d\theta$ and $J_0(x)=\frac{1}{2\pi}\int_0^{2\pi} e^{i\,x\cos\theta'}\,d\theta'$

$$\Phi(\rho=a)=\frac{1}{2\pi}\frac{1}{2\pi i}V\int_0^{2\pi}\int_0^{2\pi}e^{i\theta}\int_0^{\infty}e^{(i\cos\theta+i\cos\theta'-z/a)x}dx\,d\theta\,d\theta'$$

$$\Phi(\rho=a)=\frac{1}{2\pi}\frac{1}{2\pi}V\int_0^{2\pi}e^{i\theta}\int_0^{2\pi}\frac{1}{\cos\theta'+\cos\theta+iz/a}d\theta'd\theta$$

Many terms go away due to symmetry.

$$\Phi(\rho=a)=\frac{1}{\pi}\frac{1}{\pi}V\int_0^{\pi}\int_0^{\pi}\frac{\cos\theta\cos\theta'+\cos^2\theta}{(\cos\theta'+\cos\theta)^2+z^2/a^2}d\theta'd\theta$$

$$\Phi(\rho=a)=\frac{V}{2}\left[1-\frac{1}{\pi}[\int_0^{\pi}\frac{1}{\pi}\int_0^{\pi}\frac{\cos^2\theta'-\cos^2\theta+z^2/a^2}{(\cos\theta'+\cos\theta)^2+z^2/a^2}d\theta\,d\theta']\right]$$

$$\Phi=\frac{V}{2}\left(1-\frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{\sqrt{1+4(a/z)^2\cos^2\theta}}\right)$$

Substitute: $k=\frac{2}{\sqrt{(z/a)^2+4}}$

$$\Phi=\frac{V}{2}\left(1-\frac{kz}{\pi a}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\right)$$

$$\boxed{\Phi=\frac{V}{2}\left(1-\frac{kz}{\pi a}K(k)\right)}\quad\text{where}\quad K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

Here $K(k)$ is the complete elliptic integral of the first kind. Note that there is some ambiguity in the literature, where some sources cite the “complete elliptic integral of the first kind” as:

$$K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

and some cite it as:

$$K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k\sin^2\theta}}$$

The first definition is used here because it gives the solution that exactly matches Jackson. In the end, it should not matter which one you use as long as you are careful, consistent, and make clear which one you are using. But using the one definition when the other one is meant leads to non-physical results.