Jackson 3.10 Homework Problem Solution

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PROBLEM:

For the cylinder in Problem 3.9 the cylindrical surface is made of two equal half-cylinders, one at potential *V* and the other at potential -*V*, so that

$$V(\phi, z) = \begin{Bmatrix} V & \text{for} & -\pi/2 < \phi < \pi/2 \\ -V & \text{for} & \pi/2 < \phi < 3\pi/2 \end{Bmatrix}$$

(a) Find the potential inside the cylinder

(b) Assuming $L >> b$, consider the potential at $z = L/2$ as a function of ρ and ϕ and compare it with the two-dimensional Problem 2.13.

SOLUTION:

We simply plug in the explicit form of the boundary condition into the solution found in the previous problem:

$$C_{m,n} = \int_0^L \int_{-\pi/2}^{\pi/2} (V) e^{-im\phi} \sin\left(n\pi \frac{z}{L}\right) d\phi\, dz + \int_0^L \int_{\pi/2}^{3\pi/2} (-V) e^{-im\phi} \sin\left(n\pi \frac{z}{L}\right) d\phi\, dz$$

$$C_{m,n} = V \int_{-\pi/2}^{\pi/2} e^{-im\phi} d\phi \int_0^L \sin\left(n\pi \frac{z}{L}\right) dz - V \int_{\pi/2}^{3\pi/2} e^{-im\phi} d\phi \int_0^L \sin\left(n\pi \frac{z}{L}\right) dz$$

$$C_{m,n} = V\left[\left(\frac{1}{n\pi/L}\right)(1-(-1)^n)\right]\left[\int_{-\pi/2}^{\pi/2} e^{-im\phi} d\phi - \int_{\pi/2}^{3\pi/2} e^{-im\phi} d\phi\right]$$

$$C_{m,n} = \frac{i}{m} V \left(\frac{1}{n\pi/L}\right)(1-(-1)^n)(1-(-1)^m)^2(-1)e^{im\pi/2}$$

$$C_{m,n} = 0 \text{ if } n = \text{ even or } m = \text{ even} \quad \text{else} \quad C_{m,n} = \frac{8LV}{mn\pi}(-1)^{(m-1)/2}$$

$$\boxed{\Phi(\rho, \phi, z) = \frac{16V}{\pi^2} \sum_{m=1,\text{odd}}^{\infty} \sum_{n=1,\text{odd}}^{\infty} \frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)} \frac{1}{mn} (-1)^{(m-1)/2} \cos(m\phi) \sin(n\pi z/L)}$$

(b) Assuming $L >> b$, consider the potential at $z = L/2$ as a function of ρ and ϕ and compare it with the two-dimensional Problem 2.13.

At $z = L/2$ the potential becomes:

$$\Phi(\rho,\phi)=\frac{16V}{\pi^2}\sum_{m=1,\text{odd}}^{\infty}\sum_{n=1,\text{odd}}^{\infty}\frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\frac{1}{mn}(-1)^{(m-1)/2}\cos(m\phi)(-1)^{(n-1)/2}$$

For $L >> b$ we use the limiting form of the modified Bessel functions.

$$\Phi(\rho,\phi)=\frac{16V}{\pi^2}\sum_{m=1,\text{odd}}^{\infty}\sum_{n=1,\text{odd}}^{\infty}\frac{\rho^m}{b^m}\frac{1}{mn}(-1)^{(m-1)/2}\cos(m\phi)(-1)^{(n-1)/2}$$

$$\Phi(\rho,\phi)=\frac{16V}{\pi^2}\sum_{m=1,\text{odd}}^{\infty}\frac{\rho^m}{b^m}\frac{1}{m}(-1)^{(m-1)/2}\cos(m\phi)\sum_{n=1,\text{odd}}^{\infty}\frac{1}{n}(-1)^{(n-1)/2}$$

$$\Phi(\rho,\phi)=\frac{4V}{\pi}\sum_{m=1,\text{odd}}^{\infty}\frac{\rho^m}{b^m}\frac{1}{m}(-1)^{(m-1)/2}\cos(m\phi)$$

$$\Phi(\rho,\phi)=\frac{4V}{\pi}\Re\left[\sum_{m=1,\text{odd}}^{\infty}\left(\frac{\rho}{b}e^{i\phi}\right)^m\frac{1}{m}(-1)^{(m-1)/2}\right]$$

We recognize the expansion of the inverse tangent: $\tan^{-1}(x)=\sum_{m=1,\text{odd}}(-1)^{(m-1)/2}\frac{x^m}{m}$

$$\Phi(\rho,\phi)=\frac{4V}{\pi}\Re\left[\tan^{-1}\left(\frac{\rho}{b}e^{i\phi}\right)\right]$$

Use the identity $\Re\left[\tan^{-1}(x+iy)\right]=\frac{1}{2}\tan^{-1}\left(\frac{x}{1+y}\right)+\frac{1}{2}\tan^{-1}\left(\frac{x}{1-y}\right)$

$$\Phi(\rho,\phi)=\frac{2V}{\pi}\left[\tan^{-1}\left(\frac{\rho\cos\phi}{b+\rho\sin\phi}\right)+\tan^{-1}\left(\frac{\rho\cos\phi}{b-\rho\sin\phi}\right)\right]$$

Use the identity $\tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}$

$$\Phi(\rho,\phi)=\frac{2V}{\pi}\left[\tan^{-1}\frac{(\rho\cos\phi(b-\rho\sin\phi)+\rho\cos\phi(b+\rho\sin\phi))}{(b^2-\rho^2\sin^2\phi)-(\rho^2\cos^2\phi)}\right]$$

$$\boxed{\Phi(\rho,\phi)=\frac{2V}{\pi}\tan^{-1}\left[\frac{2b\rho\cos\phi}{b^2-\rho^2}\right]}$$

This is the same solution found in Problem 2.13 if we set $V_1 = V$ and $V_2 = -V$.