# Jackson 2.9 Homework Problem Solution

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**PROBLEM:**

An insulated, spherical, conducting shell of radius $a$ is in a uniform electric field $E_0$. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating

(a) if the shell is uncharged;

(b) if the total charge on the shell is $Q$.

**SOLUTION:**

(a) Align the $z$ axis with the direction of the electric field. Find the potential outside a sphere at the origin in a uniform field by placing charges at $z = -R$ and $z = +R$ with charges $+Q$ and $-Q$ and letting $R$ and $Q$ approach infinity with $Q/R^2$ constant. The response of the sphere can be represented by placing two image charges $-Qa/R$ and $+Qa/R$ in the sphere at $-a^2/R$ and $+a^2/R$. The potential outside an uncharged conductor in a uniform field is therefore the potential of these four charges:

$$\Phi=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+R^2+2rR\cos\theta}}-\frac{1}{\sqrt{r^2+R^2-2rR\cos\theta}}+\frac{a/R}{\sqrt{r^2+\frac{a^4}{R^2}-2a^2\frac{r}{R}\cos\theta}}-\frac{a/R}{\sqrt{r^2+\frac{a^4}{R^2}+2a^2\frac{r}{R}\cos\theta}}\right]$$

In the limit $R >> r$, this becomes:

$$\Phi=-E_0 r\cos\theta+E_0\frac{a^3}{r^2}\cos\theta$$ where $E_0$ was recognized as $2Q/4\pi\varepsilon_0 R^2$

The first term is just the potential due to the applied field in spherical coordinates. The second term is the potential of a perfect dipole. The sphere there has an induced charge distribution that acts as a perfect dipole.

The electric field is therefore:

$$\mathbf{E}=-\nabla\Phi$$

$$\mathbf{E}=E_0\left[\cos\theta\,\hat{\mathbf{r}}-\sin\theta\,\hat{\boldsymbol{\theta}}+\frac{a^3}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)\right]$$

The electric field at the surface of the sphere is:

$$\mathbf{E}(r=a)=E_0 3\cos\theta\,\hat{\mathbf{r}}$$

The charge distribution on the sphere's surface is found using:

$$\sigma = \epsilon_0 \hat{\mathbf{r}} \cdot \mathbf{E}(r=a)$$

$$\sigma = 3\epsilon_0 E_0 \cos\theta$$

If the sphere is now cut into hemispheres at the polar angle $\theta = \pi/2$, the bottom hemisphere will feel a total force:

$$\mathbf{F} = \int \sigma(\mathbf{x}) \mathbf{E}(\mathbf{x}) \, d\,a$$

We have to be careful and not include the force of the bottom hemisphere on itself. We do this by using the relation $E = \frac{\sigma}{2\epsilon_0}$ which gives us just the electric field at the surface of a conductor due to non-self contributions. Using this, we have:

$$\mathbf{F} = \frac{1}{2\epsilon_0} \int \sigma^2 \hat{\mathbf{r}} \, d\,a$$

$$\mathbf{F} = \hat{\mathbf{k}} \frac{a^2}{2\epsilon_0} \int_0^{2\pi} \int_{\pi/2}^{\pi} (3\epsilon_0 E_0 \cos\theta)^2 \cos\theta \sin\theta \, d\theta \, d\phi$$

$$\mathbf{F} = \hat{\mathbf{k}}\, 9\epsilon_0 \pi E_0^2 a^2 \int_{\pi/2}^{\pi} \cos^3\theta \sin\theta \, d\theta$$

$$\mathbf{F} = -\frac{9}{4} \pi \epsilon_0 E_0^2 a^2 \hat{\mathbf{k}}$$

The force needed to keep the bottom hemisphere in place would therefore have to be equal and in the opposite direction:

$$\boxed{\mathbf{F} = \frac{9}{4} \pi \epsilon_0 E_0^2 a^2 \hat{\mathbf{k}}}$$

Due the symmetry, the force needed to keep the other hemisphere in place would be equal and opposite.

(b) If the sphere has a total charge of $Q$, it will just spread out uniformly on the sphere as an additional charge to the induced one.

$$\sigma = 3\epsilon_0 E_0 \cos\theta + \frac{Q}{4\pi a^2}$$

The total force on the bottom hemisphere will therefore be:

$$\mathbf{F} = \frac{1}{2\epsilon_0} \int \sigma^2 \hat{\mathbf{r}} \, d\,a$$

$$\mathbf{F}=\hat{\mathbf{k}}\frac{a^2}{2\epsilon_0}\int_0^{2\pi}\int_{\pi/2}^{\pi}(3\epsilon_0 E_0\cos\theta+\frac{Q}{4\pi a^2})(3\epsilon_0 E_0\cos\theta+\frac{Q}{4\pi a^2})\cos\theta\sin\theta\, d\theta\, d\phi$$

$$\mathbf{F}=\frac{a^2}{2\epsilon_0}\hat{\mathbf{k}}\int_0^{2\pi}\int_{\pi/2}^{\pi}(9\epsilon_0^2 E_0^2\cos^2\theta+6\epsilon_0 E_0\cos\theta\frac{Q}{4\pi a^2}+\frac{Q^2}{16\pi^2 a^4})\cos\theta\sin\theta\, d\theta\, d\phi$$

The first term represents the force on the induced charges due the external field and the field from the induced charges. The second term represents the force on the charge $Q$ due the external field. The third term represent the force $Q$ on itself. Note that the force of the external field on the point-charge-like charge $Q$ will just tend to shift it and not separate it. Because we just want forces that will separate the two hemispheres, we must drop the middle term:

$$\mathbf{F}=\frac{a^2}{2\epsilon_0}\hat{\mathbf{k}}\int_0^{2\pi}\int_{\pi/2}^{\pi}(9\epsilon_0^2 E_0^2\cos^2\theta+\frac{Q^2}{16\pi^2 a^4})\cos\theta\sin\theta\, d\theta\, d\phi$$

$$\mathbf{F}=-\hat{\mathbf{k}}\left[\frac{9}{4}\pi a^2\epsilon_0 E_0^2+\frac{Q^2}{32\pi\epsilon_0 a^2}\right]$$

The force needed to keep the bottom hemisphere touching the upper sphere is therefore:

$$\boxed{\mathbf{F}=\hat{\mathbf{k}}\left[\frac{9}{4}\pi a^2\epsilon_0 E_0^2+\frac{Q^2}{32\pi\epsilon_0 a^2}\right]}$$