Jackson 2.7 Homework Problem Solution

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PROBLEM:
Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

(a) Write down the appropriate Green function G(**x**, **x**')

(b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered on the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).

(c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

(d) Show that at large distances ($\rho^2 + z^2 >> a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{V a^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2+z^2)^2} + \ldots\right]$$

Verify that the results of parts c and d are consistent with each other in their common range of validity.

SOLUTION:
(a) The Green function solution to the potential problem with this boundary for Dirichlet boundary conditions requires that $G_D = 0$ on the surface and that G_D obeys

$$\psi = G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}')$$

everywhere else. The problem of finding the Green function amounts to finding the proper function F above so that the Green function G disappears on the boundary. This problem is exactly equivalent to a the situation where we have a unit point charge at **x'**, which creates the potential 1/|**x**-**x'**|, in the presence of a flat conductor running along the $z = 0$ plane. The solution for the potential of the equivalent problem will then be identical to the Green function for the original problem.

To solve the problem of the charge q at **x'** in the presence of a conducting sheet at the $z = 0$ plane, we use the method of images. We place an image charge q' at (x', y', $-z'$) so that the potential is just the sum of the two point charges:

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{4\pi\epsilon_0}\frac{q'}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

Apply the boundary condition of a conductor at the $z = 0$ plane: $\Phi(x, y, z=0)=0$

$$0 = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z')^2}} + \frac{1}{4\pi\epsilon_0}\frac{q'}{\sqrt{(x-x')^2+(y-y')^2+(z')^2}}$$

$$q' = -q$$

The image charge is just the perfect mirror image in location and in charge, which makes sense because a perfectly conducting surface that is perfectly flat is just a perfect mirror. The potential is then:

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]$$

Now go back to the original problem without the conductor and point, but some potentially complex charge distribution in the presence of some potentially complex boundary condition along the $z = 0$ plane. The solution to this original problem is, according to the Green function method:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\, G_D\, d^3\mathbf{x}' - \frac{1}{4\pi}\oint\left(\Phi\frac{d\,G_D}{d\,n'}\right) da'$$

where we now know the Green function used by this equation. It is just the solution to the equivalent problem with unit charge ($q = 4\pi\varepsilon_0$):

$$\boxed{G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}}$$

(b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered on the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).

Although not stated, it can be assumed that there is no charge present anywhere in this problem. when that is the case, then the Green function solution for Dirichlet boundary conditions becomes:

$$\Phi(\mathbf{x}) = \frac{-1}{4\pi}\oint_S\left(\Phi\frac{d\,G_D}{d\,n'}\right) da'$$

The enclosing surface S in this case is a box with one side on the plane $z = 0$ and the other sides at infinity. The potential dies off to zero at infinity, so the sides at infinity make no contribution to the integral. The potential is also zero everywhere on the plane $z = 0$ outside the circle, so that those locations make no contribution to the integral. The only piece of the integral left is inside the circle, so that:

$$\Phi(\mathbf{x})=\frac{-1}{4\pi}\int_0^a\int_0^{2\pi}\left(\Phi\frac{d\,G_D}{d\,n'}\right)\rho'\,d\,\phi'\,d\,\rho'$$

$$\Phi(\mathbf{x})=\frac{-V}{4\pi}\int_0^a\int_0^{2\pi}\frac{d\,G_D}{d\,n'}\rho'\,d\,\phi'\,d\,\rho'$$

The normal *n*' is defined as pointing out of the volume enclosed so that for this problem *n*' = -*z*'

$$\Phi(\mathbf{x})=\frac{V}{4\pi}\int_0^a\int_0^{2\pi}\frac{d\,G_D}{d\,z'}\rho'\,d\,\phi'\,d\,\rho'$$

Plugging in the Green function we found above:

$$\Phi(\mathbf{x})=\frac{V}{4\pi}\int_0^a\int_0^{2\pi}\frac{d}{d\,z'}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]\rho'\,d\,\phi'\,d\,\rho'$$

Putting everything in cylindrical coordinates:

$$\Phi(\mathbf{x})=\frac{V}{4\pi}\int_0^a\int_0^{2\pi}\frac{d}{d\,z'}[\frac{1}{\sqrt{\rho^2+\rho'^2-2\,\rho\,\rho'\cos(\phi-\phi')+(z-z')^2}}-\frac{1}{\sqrt{\rho^2+\rho'^2-2\,\rho\,\rho'\cos(\phi-\phi')+(z+z')^2}}]\rho'\,d\,\phi'\,d\,\rho'$$

$$\Phi(\mathbf{x})=\frac{V}{4\pi}\int_0^a\int_0^{2\pi}[(-1/2)\frac{2(z-z')(-1)}{(\rho^2+\rho'^2-2\,\rho\,\rho'\cos(\phi-\phi')+(z-z')^2)^{3/2}}-(-1/2)\frac{2(z+z')}{(\rho^2+\rho'^2-2\,\rho\,\rho'\cos(\phi-\phi')+(z+z')^2)^{3/2}}]\rho'\,d\,\phi'\,d\,\rho'$$

We now evaluate everything at $z'=0$ because our integration surface (the primed system) is completely contained in this plane.

$$\Phi(\mathbf{x})=\frac{z\,V}{2\pi}\int_0^a\int_0^{2\pi}\frac{\rho'\,d\,\phi'\,d\,\rho'}{(\rho^2+\rho'^2-2\,\rho\,\rho'\cos(\phi-\phi')+z^2)^{3/2}}$$

The problem is azimuthally symmetric, so that we are free to make a change of variables:
$\phi'\rightarrow\phi'+\phi$

$$\boxed{\Phi(\mathbf{x})=\frac{z\,V}{2\pi}\int_0^a\int_0^{2\pi}\frac{\rho'\,d\,\phi'\,d\,\rho'}{(\rho^2+\rho'^2-2\,\rho\,\rho'\cos(\phi')+z^2)^{3/2}}}$$

(c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Take the general solution found above and plug in ($\rho = 0$) to get the on-axis solution:

$$\Phi(\mathbf{x}) = \frac{zV}{2\pi}\int_0^a\int_0^{2\pi}\frac{\rho' d\phi' d\rho'}{(\rho'^2+z^2)^{3/2}}$$

$$\Phi(\mathbf{x}) = \frac{zV}{2\pi}\int_0^{2\pi} d\phi' \int_0^a \frac{\rho' d\rho'}{(\rho'^2+z^2)^{3/2}}$$

$$\Phi(\mathbf{x}) = zV\int_0^a \frac{\rho' d\rho'}{(\rho'^2+z^2)^{3/2}}$$

Make a change of variables $u = \rho'^2 + z^2$, $du = 2\rho' d\rho'$

$$\Phi(\mathbf{x}) = \frac{zV}{2}\int_{z^2}^{a^2+z^2}\frac{du}{u^{3/2}}$$

$$\Phi(\mathbf{x}) = -zV\left[\frac{1}{\sqrt{u}}\right]_{z^2}^{a^2+z^2}$$

$$\boxed{\Phi(\mathbf{x}) = V\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]}$$

(d) Show that at large distances ($\rho^2 + z^2 >> a^2$) the potential can be expanded in a power series in ($\rho^2 + z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2} + \ldots\right]$$

Verify that the results of parts c and d are consistent with each other in their common range of validity.

The general solution found above was:

$$\Phi(\mathbf{x}) = \frac{zV}{2\pi}\int_0^a\int_0^{2\pi}\frac{\rho' d\phi' d\rho'}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi')+z^2)^{3/2}}$$

Divide top and bottom by $(\rho^2+z^2)^{3/2}$

$$\Phi(\mathbf{x})=\frac{zV}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\left(1+\frac{\rho'^2-2\rho\rho'\cos(\phi')}{\rho^2+z^2}\right)^{-3/2}$$

Expand the last factor using the Binomial series: $(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+...$

$$\Phi(\mathbf{x})=\frac{zV}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'[1-\frac{3}{2}(\rho^2+z^2)^{-1}(\rho'^2-2\rho\rho'\cos(\phi'))$$
$$+\frac{15}{8}(\rho^2+z^2)^{-2}(\rho'^2-2\rho\rho'\cos(\phi'))^2+...]$$

We can now integrate term by term

$$\Phi(\mathbf{x})=\frac{zV}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}[\left[\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\right]-\frac{3}{2}(\rho^2+z^2)^{-1}\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\left[\rho'^2-2\rho\rho'\cos(\phi')\right]$$
$$+\frac{15}{8}(\rho^2+z^2)^{-2}\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\left[(\rho'^2-2\rho\rho'\cos(\phi'))^2\right]+...]$$

$$\Phi(\mathbf{x})=\frac{zV}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}[\pi a^2-\frac{3}{2}(\rho^2+z^2)^{-1}\left[\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\left[\rho'^2\right]-2\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\left[\rho\rho'\cos(\phi')\right]\right]$$
$$+\frac{15}{8}(\rho^2+z^2)^{-2}\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\left[\rho'^4+4\rho^2\rho'^2\cos^2(\phi')-4\rho\rho'^3\cos(\phi')\right]+...]$$

$$\Phi(\mathbf{x})=\frac{zV}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}[\pi a^2-\frac{3}{2}(\rho^2+z^2)^{-1}\left[\frac{\pi}{2}a^4\right]+\frac{15}{8}(\rho^2+z^2)^{-2}\left[2\pi\frac{a^6}{6}+4\rho^2\frac{a^4}{4}\pi\right]+...]$$

$$\boxed{\Phi(\mathbf{x})=\frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2}+...\right]}$$

Along the axis this becomes

$$\Phi(\mathbf{x})=\frac{Va^2}{2}\frac{1}{z^2}\left[1-\frac{3a^2}{4z^2}+\frac{5a^4}{8z^4}+...\right]$$

$$\Phi(\mathbf{x})=V\left[\frac{a^2}{2z^2}-\frac{3a^4}{8z^4}+\frac{5a^6}{16z^6}+...\right]$$

$$\Phi(\mathbf{x})=V\left[1-\left(1-\frac{a^2}{2z^2}+\frac{3a^4}{8z^4}-\frac{5a^6}{16z^6}+...\right)\right]$$

Now we recognize the expansion: $(1+x)^{-1/2}=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3$

$$\Phi(\mathbf{x}) = V\left[1-\left(1+\frac{a^2}{z^2}\right)^{-1/2}\right]$$

$$\boxed{\Phi(\mathbf{x}) = V\left[1-\frac{z}{\sqrt{a^2+z^2}}\right]}$$

This matches part c.