Jackson 2.5 Homework Problem Solution

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PROBLEM:

(a) Show that the work done to remove the charge q from a distance $r > a$ to infinity against the force, Eq. (2.6), of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)}$$

Relate this result to the electrostatic potential, Eq. (2.3), and the energy discussion of Section 1.11.

(b) Repeat the calculation of the work done to remove the charge q against the force, Eq. (2.9), of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

Relate the work to the electrostatic potential, Eq. (2.8), and the energy discussion of Section 1.11.

SOLUTION:

(a) A charge q near a grounded conducing sphere feels an attractive force:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1 - \frac{a^2}{y^2}\right)^{-2}$$

The work done to remove it to infinity is:

$$W = -\int_A^B \mathbf{F}\cdot d\,\mathbf{l}$$

$$W = \int_r^\infty \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1 - \frac{a^2}{y^2}\right)^{-2} dy$$

$$W = \frac{1}{4\pi\epsilon_0} q^2 a \int_r^\infty \frac{y}{(y^2 - a^2)^2} dy$$

Use $u = y^2 - a^2$, $d\,u = 2\,y\,dy$

$$W = \frac{1}{8\pi\epsilon_0} q^2 a \int_{r^2 - a^2}^\infty \frac{du}{u^2}$$

$$\boxed{W=\frac{q^2 a}{8\pi\epsilon_0 (r^2-a^2)}}$$

The work is the charge times potential difference, or just the potential if we have a zero-potential reference point at infinity, as we do here. We found the potential to be:

$$\Phi(\mathbf{x})=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|}-\frac{1}{\left|\frac{y}{a}\mathbf{x}-\frac{a}{y}\mathbf{y}\right|}\right]$$

In this case, the charge helping to create the potential is the same one we are doing work on, so that **x** = **y** = **r**. (The infinite self-energy of the first term is non-physical and is dropped).

$$q\Phi(\mathbf{x})=\frac{-q^2}{4\pi\epsilon_0}\frac{a}{(r^2-a^2)}$$

Comparing this expression to the work expression, we see they are identical except for a few features. First, the sign difference accounts for the fact that we have removed the particle against a force. Also, the missing factor of one half is accounted by the fact that we have double counted when the charge helping to the create the potential is the same one that is having work done on it against the potential.

(b) To remove a charge from near an isolated, charged, conducting sphere, we do the same type of calculation:

$$F=\frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\left[Q-\frac{q a^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]$$

$$W=\frac{-q}{4\pi\epsilon_0}\int_r^\infty\frac{1}{y^2}\left[Q-\frac{q a^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]dy$$

$$W=\frac{-q}{4\pi\epsilon_0}\left[\frac{Q}{r}-q a^3\int_r^\infty\frac{(2y^2-a^2)}{y^3(y^2-a^2)^2}dy\right]$$

Use $u=y^2-a^2$, $du=2y\,dy$, $y^2=u+a^2$ and separate into partial fractions:

$$W=\frac{-q}{4\pi\epsilon_0}\left[\frac{Q}{r}-\frac{qa}{2}\left[-\int_{r^2-a^2}^\infty\frac{1}{(u+a^2)^2}du+\int_{r^2-a^2}^\infty\frac{1}{u^2}du\right]\right]$$

$$\boxed{W=\frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2-a^2)}-\frac{q^2 a}{2r^2}-\frac{qQ}{r}\right]}$$

The potential when **x** = **y** = **r** is:

$$\boxed{q\Phi = \frac{-1}{4\pi\epsilon_0}\left[\frac{q^2 a}{(r^2 - a^2)} - \frac{q^2 a}{r^2} - \frac{qQ}{r}\right]}$$

Comparing this expression to the work expression, we see that they match apart from the overall sign and a factor of one half. Notice that the factor of one half only exists on the pieces related to the original charge, because this is the only one that gets double-counted.