PROBLEM:

A point charge is placed a distance $d > R$ from the center of an equally charged, isolated, conducting sphere or radius R.

(a) Inside of what distance from the surface of the sphere is the point charge attracted rather than repelled by the charged sphere?

(b) What is the limiting value of the force of attraction when the point charge is located a distance $a\ (= d - R)$ from the surface of the sphere, if $a << R$?

(c) What are the results for parts a and b if the charge on the sphere is twice (half) as large as the point charge, but still the same sign?

SOLUTION:

We can treat the final solution as the sum of a grounded conducting sphere plus the remaining charge on the sphere which spreads out uniformly. Call the total charge on the point charge q and the total charge on the sphere Q. If the sphere were grounded, it would have a total charge induced on it equal to the image charge q'. The remaining charge $Q - q'$ spreads out uniformly and creates a potential equivalent to a point charge at its center. The total potential is then the sum of that due to the original charge, the image charge, and the remaining charge (note that $d = |\mathbf{x}_0|$ designates the position of the point charge):

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{x}_0|} - \frac{Rq}{d\left|\mathbf{x}-\frac{R^2}{d^2}\mathbf{x}_0\right|} + \frac{Q+\frac{R}{d}q}{|\mathbf{x}|}\right]$$

(a) This potential is valid at all points in space. Because all we care about is the potential which leads to a force on the point charge, we can constrain our points of interest to the axis of the charge-sphere system, $\hat{\mathbf{x}} = \hat{\mathbf{x}}_0$, and also drop the first term because the particle does not exert a total force on itself: (Note that $r = |\mathbf{x}|$).

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{-Rq}{rd-R^2} + \frac{Q}{r} + \frac{Rq}{rd}\right]$$

The force the point charge experiences is:

$$\mathbf{F} = q\,\mathbf{E}(r=d)$$

$$F = \left[-q\,\nabla\,\Phi \right]_{r=d}$$

$$F = \hat{\mathbf{r}}\left[-q\,\frac{\partial\,\Phi}{\partial r} \right]_{r=d}$$

$$F = \hat{\mathbf{r}}\,\frac{-q}{4\,\pi\,\epsilon_0}\left[\frac{R\,q\,d}{(r\,d - R^2)^2} - \frac{Q}{r^2} - \frac{R\,q}{r^2\,d} \right]_{r=d}$$

$$\boxed{F = \hat{\mathbf{r}}\,\frac{q^2}{4\,\pi\,\epsilon_0\,d^2}\left[\frac{-\left(\dfrac{d}{R}\right)^3}{\left(\left(\dfrac{d}{R}\right)^2 - 1\right)^2} + Q/q + \frac{R}{d} \right]}$$

The point of turning from being repelled to attracted occurs when the force equals zero:

$$0 = \frac{-\left(\dfrac{d}{R}\right)^3}{\left(\left(\dfrac{d}{R}\right)^2 - 1\right)^2} + Q/q + \frac{R}{d}$$

$$\left(\frac{d}{R}\right)^3 = \left(Q/q + \frac{R}{d}\right)\left(\left(\frac{d}{R}\right)^2 - 1\right)^2$$

$$\boxed{(Q/q)\left(\frac{d}{R}\right)^5 - 2(Q/q)\left(\frac{d}{R}\right)^3 - 2\left(\frac{d}{R}\right)^2 + \left(\frac{d}{R}\right)(Q/q) + 1 = 0}$$

For $Q = q$:

$$\left(\frac{d}{R}\right)^5 - 2\left(\frac{d}{R}\right)^3 - 2\left(\frac{d}{R}\right)^2 + \left(\frac{d}{R}\right) + 1 = 0$$

This is solved numerically to give: $\boxed{\dfrac{d}{R} = 1.61803}$

(b) When the point charge is close to the surface, $d = R + a$ and $a \ll R$:

$$F = \hat{\mathbf{r}}\,\frac{q^2}{4\,\pi\,\epsilon_0\,(R+a)^2}\left[\frac{-\left(\dfrac{(R+a)}{R}\right)^3}{\left(\left(\dfrac{(R+a)}{R}\right)^2 - 1\right)^2} + Q/q + \frac{R}{(R+a)} \right]$$

$$\mathbf{F}=\hat{\mathbf{r}}\frac{q^2}{4\pi\epsilon_0}\left[\frac{-R(R+a)^4+a^2(R\,a^2+4\,R^3+4\,a\,R^2+Q/q(5\,R\,a^2+4\,R^3+8\,a\,R^2+a^3))}{a^2(a^2+4\,R^2+4\,a\,R)(R+a)^3}\right]$$

Now apply $a \ll R$:

$$\mathbf{F}=\hat{\mathbf{r}}\frac{q^2}{4\pi\epsilon_0}\left[\frac{-R^5}{a^2(4\,R^5)}\right]$$

$$\boxed{\mathbf{F}=-\hat{\mathbf{r}}\frac{q^2}{16\pi\epsilon_0 a^2}}$$

This is the force between a point charge and its image in an infinite plane. This means that very close to the sphere, it looks like a plane, and the excess charge is far away enough to have no effect.

(c) The results for part (b) are general and will still be the same no matter what the total charge on the sphere is. The turning point of part (a) will however change.

For $Q = 2q$ we must solve:

$$2\left(\frac{d}{R}\right)^5-4\left(\frac{d}{R}\right)^3-2\left(\frac{d}{R}\right)^2+2\left(\frac{d}{R}\right)+1=0$$

Using a numerical solver, we find: $\boxed{\dfrac{d}{R}=1.42756}$

For $Q = q/2$ we must solve:

$$1/2\left(\frac{d}{R}\right)^5-\left(\frac{d}{R}\right)^3-2\left(\frac{d}{R}\right)^2+\left(\frac{d}{R}\right)1/2+1=0$$

Using a numerical solver, we find: $\boxed{\dfrac{d}{R}=1.88227}$