Jackson 2.3 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell

PROBLEM:

A straight-line charge with constant linear charge λ is located perpendicular to the x-y plane in the first quadrant at (x_0, y_0). The intersecting planes at $x = 0, y \geq 0$ and $y = 0, x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

(a) The well-known potential for an isolated line charge at (x_0, y_0) is $\Phi(x, y) = (\lambda/4\pi\varepsilon_0)\ln(R^2/r^2)$, where $r^2 = (x - x_0)^2 + (y - y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish at the boundary surfaces.

(b) Determine the surface charge density σ on the plane $y = 0, x \geq 0$. Plot σ/λ versus x for $(x_0 = 2, y_0 = 1)$, $(x_0 = 1, y_0 = 1)$, $(x_0 = 1, y_0 = 2)$.

(c) Show that the total charge (per unit length in z) on the plane $y = 0, x \geq 0$ is

$$Q_x = -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right)$$

What is the total charge on the plane $x = 0$?

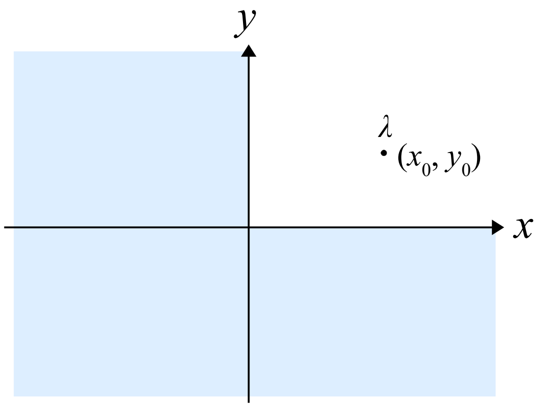
(d) Show that far from the origin [$\rho \gg \rho_0$, where $\rho = \sqrt{x^2 + y^2}$ and $\rho_0 = \sqrt{x_0^2 + y_0^2}$] the leading term in the potential is

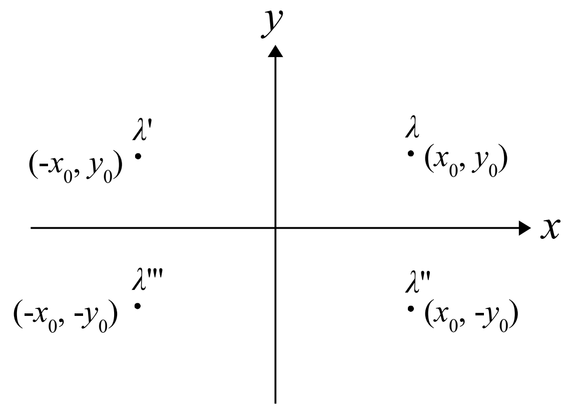
$$\Phi \rightarrow \Phi_{\text{asym}} = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0 y_0)(xy)}{\rho^4}$$

Interpret.

SOLUTION:

Using the method of images, let us put an image line charge λ' at $(-x_0, y_0)$, an image line charge λ'' at $(x_0, -y_0)$, and an image line charge λ''' at $(-x_0, -y_0)$ and conceptually remove the conducting surface.





The solution to the potential for the four line charges is:

$$\Phi(x,y)=\frac{1}{4\pi\epsilon_0}\lambda\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right)+\frac{1}{4\pi\epsilon_0}\lambda'\ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)$$
$$+\frac{1}{4\pi\epsilon_0}\lambda''\ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right)+\frac{1}{4\pi\epsilon_0}\lambda'''\ln\left(\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right)$$

Apply the boundary condition $\Phi(x=0,y)=0$

$$0=\lambda\ln\left(\frac{R^2}{x_0^2+(y-y_0)^2}\right)+\lambda'\ln\left(\frac{R^2}{x_0^2+(y-y_0)^2}\right)+\lambda''\ln\left(\frac{R^2}{x_0^2+(y+y_0)^2}\right)+\lambda'''\ln\left(\frac{R^2}{x_0^2+(y+y_0)^2}\right)$$

$$0=\ln\left[\left(\frac{R^2}{x_0^2+(y-y_0)^2}\right)^{\lambda}\left(\frac{R^2}{x_0^2+(y-y_0)^2}\right)^{\lambda'}\left(\frac{R^2}{x_0^2+(y+y_0)^2}\right)^{\lambda''}\left(\frac{R^2}{x_0^2+(y+y_0)^2}\right)^{\lambda'''}\right]$$

$$1=\left(\frac{R^2}{x_0^2+(y-y_0)^2}\right)^{\lambda+\lambda'}\left(\frac{R^2}{x_0^2+(y+y_0)^2}\right)^{\lambda''+\lambda'''}$$

This can only be true for all y if $\boxed{\lambda+\lambda'=0}$ and $\boxed{\lambda''+\lambda'''=0}$

Apply the boundary condition $\Phi(x,y=0)=0$

$$0=\lambda\ln\left(\frac{R^2}{(x-x_0)^2+y_0^2}\right)+\lambda'\ln\left(\frac{R^2}{(x+x_0)^2+y_0^2}\right)+\lambda''\ln\left(\frac{R^2}{(x-x_0)^2+y_0^2}\right)+\lambda'''\ln\left(\frac{R^2}{(x+x_0)^2+y_0^2}\right)$$

$$1=\left(\frac{R^2}{(x-x_0)^2+y_0^2}\right)^{\lambda+\lambda''}\left(\frac{R^2}{(x+x_0)^2+y_0^2}\right)^{\lambda'+\lambda'''}$$

This can only be true for all x if $\boxed{\lambda+\lambda''=0}$ and $\boxed{\lambda'+\lambda'''=0}$

Using the four equations in boxes, we now have four equations and three unknowns. We solve for each:

$$\lambda' = -\lambda \quad , \quad \lambda'' = -\lambda \quad , \quad \lambda''' = \lambda$$

The final solution is then:

$$\Phi(x,y) = \frac{1}{4\pi\epsilon_0}\lambda \ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right) - \frac{1}{4\pi\epsilon_0}\lambda \ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)$$
$$- \frac{1}{4\pi\epsilon_0}\lambda \ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right) + \frac{1}{4\pi\epsilon_0}\lambda \ln\left(\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right)$$

$$\boxed{\Phi(x,y) = \frac{-\lambda}{4\pi\epsilon_0}[\ln\left((x-x_0)^2+(y-y_0)^2\right) - \ln\left((x+x_0)^2+(y-y_0)^2\right) - \ln\left((x-x_0)^2+(y+y_0)^2\right) + \ln\left((x+x_0)^2+(y+y_0)^2\right)]}$$

To explicitly verify that the potential disappears at the boundary, we check the potential at $x = 0$

$$\Phi(x,y) = \frac{-\lambda}{4\pi\epsilon_0}[\ln\left((x_0)^2+(y-y_0)^2\right) - \ln\left((x_0)^2+(y-y_0)^2\right) - \ln\left((x_0)^2+(y+y_0)^2\right) + \ln\left((x_0)^2+(y+y_0)^2\right)]$$

$$\boxed{\Phi(x=0,y) = 0}$$

and check the potential at $y = 0$

$$\Phi(x,y) = \frac{-\lambda}{4\pi\epsilon_0}[\ln\left((x-x_0)^2+(y_0)^2\right) - \ln\left((x+x_0)^2+(y_0)^2\right) - \ln\left((x-x_0)^2+(y_0)^2\right) + \ln\left((x+x_0)^2+(y_0)^2\right)]$$

$$\boxed{\Phi(x,y=0) = 0}$$

The tangential electric field along the x-axis boundary is just E_x:

$$E_x = -\frac{\partial \Phi}{\partial x} \quad \text{at } y = 0$$

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\frac{\partial}{\partial x}[\ln\left((x-x_0)^2+(y-y_0)^2\right) - \ln\left((x+x_0)^2+(y-y_0)^2\right)$$
$$- \ln\left((x-x_0)^2+(y+y_0)^2\right) + \ln\left((x+x_0)^2+(y+y_0)^2\right)]$$

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2} - \frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2}\right]$$

At $y = 0$:

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2(x-x_0)}{(x-x_0)^2+y_0^2} - \frac{2(x+x_0)}{(x+x_0)^2+y_0^2} - \frac{2(x-x_0)}{(x-x_0)^2+y_0^2} + \frac{2(x+x_0)}{(x+x_0)^2+y_0^2}\right]$$

$\boxed{E_x = 0}$ at $y = 0$:

The tangential electric field along the y-axis boundary is just E_y:

$$E_y = -\frac{\partial \Phi}{\partial y} \quad \text{at } x = 0$$

$$E_y = \frac{\lambda}{4\pi\epsilon_0}\frac{\partial}{\partial y}[\ln\left((x-x_0)^2+(y-y_0)^2\right) - \ln\left((x+x_0)^2+(y-y_0)^2\right)$$
$$-\ln\left((x-x_0)^2+(y+y_0)^2\right) + \ln\left((x+x_0)^2+(y+y_0)^2\right)]$$

$$E_y = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(y-y_0)}{(x+x_0)^2+(y-y_0)^2} - \frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{2(y+y_0)}{(x+x_0)^2+(y+y_0)^2}\right]$$

At $x = 0$:

$$E_y = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2(y-y_0)}{x_0^2+(y-y_0)^2} - \frac{2(y-y_0)}{x_0^2+(y-y_0)^2} - \frac{2(y+y_0)}{x_0^2+(y+y_0)^2} + \frac{2(y+y_0)}{x_0^2+(y+y_0)^2}\right]$$

$\boxed{E_y = 0}$ at $x = 0$

(b) Determine the surface charge density σ on the plane $y = 0$, $x \geq 0$. Plot σ/λ versus x for ($x_0 = 2$, $y_0 = 1$), ($x_0 = 1$, $y_0 = 1$), ($x_0 = 1$, $y_0 = 2$).

The surface charge density on an arbitrary surface creates an electric field discontinuity according to:

$$\left[(\mathbf{E}_2 - \mathbf{E}_1)\cdot\mathbf{n} = \frac{1}{\epsilon_0}\sigma\right]_{n=n_0}$$

For a conductor, the electric field below the surface is zero, $\mathbf{E}_1 = 0$, and the electric field is normal to the conductor's surface, and thus parallel to the conductor's normal, so that:

$$\left[E_n = \frac{1}{\epsilon_0}\sigma\right]_{n=n_0}$$

For this particular problem, the surface is the x-axis so that the normal is in the y direction

$$\sigma = \left[\epsilon_0 E_y\right]_{y=0}$$

We have already derived E_y above and plug it directly in:

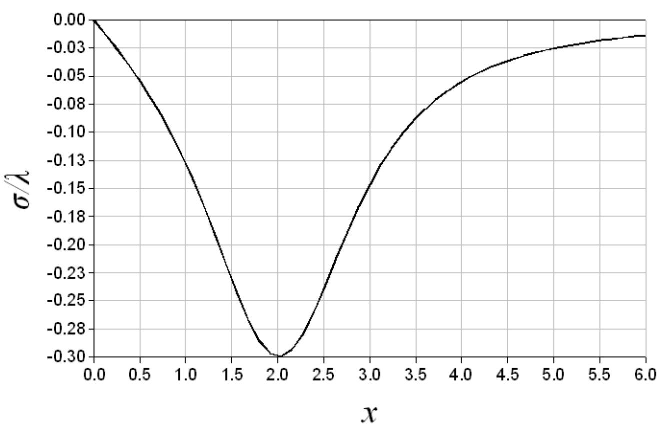
$$\sigma = \left[\frac{\lambda}{4\pi} \left[\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(y-y_0)}{(x+x_0)^2+(y-y_0)^2} - \frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{2(y+y_0)}{(x+x_0)^2+(y+y_0)^2} \right] \right]_{y=0}$$

$$\sigma = \frac{\lambda}{4\pi} \left[-\frac{4(y_0)}{(x-x_0)^2+y_0^2} + \frac{4(y_0)}{(x+x_0)^2+y_0^2} \right]$$

$$\boxed{\sigma = -\frac{\lambda y_0}{\pi} \left[\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2} \right]}$$

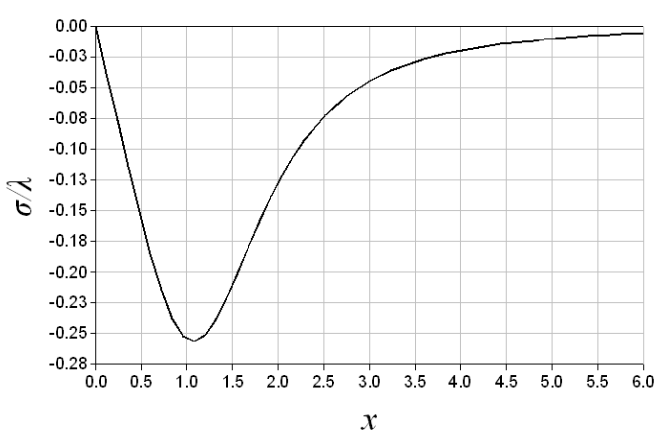
For ($x_0 = 2$, $y_0 = 1$)

$$\frac{\sigma}{\lambda} = -\frac{1}{\pi} \left[\frac{1}{(x-2)^2+1} - \frac{1}{(x+2)^2+1} \right]$$



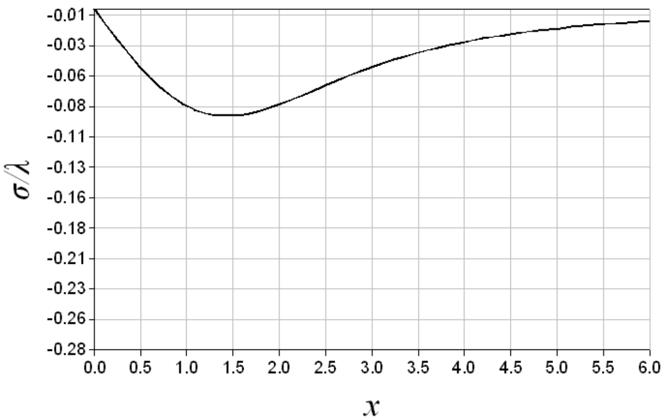
For ($x_0 = 1$, $y_0 = 1$)

$$\frac{\sigma}{\lambda} = -\frac{1}{\pi} \left[\frac{1}{(x-1)^2+1} - \frac{1}{(x+1)^2+1} \right]$$



For ($x_0 = 1, y_0 = 2$)

$$\frac{\sigma}{\lambda} = -\frac{2}{\pi}\left[\frac{1}{(x-1)^2+4} - \frac{1}{(x+1)^2+4}\right]$$



(c) Show that the total charge (per unit length in z) on the plane $y = 0, x \geq 0$ is

$$Q_x = -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right)$$

What is the total charge on the plane $x = 0$?

To get the total charge on the plane $y = 0$ we just integrate over the charge density on the plane:

$$Q_x = \int_0^\infty \sigma(x)\, dx$$

$$Q_x = -\frac{\lambda y_0}{\pi}\left[\int_0^\infty \frac{1}{(x-x_0)^2+y_0^2} dx - \int_0^\infty \frac{1}{(x+x_0)^2+y_0^2} dx\right]$$

$$Q_x = -\frac{\lambda y_0}{\pi}\left[\int_{-x_0}^\infty \frac{1}{x^2+y_0^2} dx - \int_{x_0}^\infty \frac{1}{x^2+y_0^2} dx\right]$$

$$Q_x = -\frac{\lambda y_0}{\pi}\left[\left[\frac{1}{y_0}\tan^{-1}\left(\frac{x}{y_0}\right)\right]_{-x_0}^\infty - \left[\frac{1}{y_0}\tan^{-1}\left(\frac{x}{y_0}\right)\right]_{x_0}^\infty\right]$$

$$Q_x = -\frac{\lambda y_0}{\pi}\left[\frac{\pi}{2y_0} - \frac{1}{y_0}\tan^{-1}\left(\frac{-x_0}{y_0}\right) - \frac{\pi}{2y_0} + \frac{1}{y_0}\tan^{-1}\left(\frac{x_0}{y_0}\right)\right]$$

$$\boxed{Q_x = -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right)}$$

Due to the total symmetry between the *x* and *y* axes, the total charge on the *x* = 0 plane is:

$$\boxed{Q_x = -\frac{2}{\pi}\lambda\tan^{-1}\left(\frac{y_0}{x_0}\right)}$$

(d) Show that far from the origin [$\rho \gg \rho_0$, where $\rho = \sqrt{x^2+y^2}$ and $\rho_0 = \sqrt{x_0^2+y_0^2}$] the leading term in the potential is

$$\Phi \to \Phi_{\text{asym}} = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0 y_0)(xy)}{\rho^4}$$

Interpret.

The potential was found above to be:

$$\Phi(x,y) = \frac{-\lambda}{4\pi\epsilon_0}[\ln\left((x-x_0)^2+(y-y_0)^2\right) - \ln\left((x+x_0)^2+(y-y_0)^2\right) \\ -\ln\left((x-x_0)^2+(y+y_0)^2\right) + \ln\left((x+x_0)^2+(y+y_0)^2\right)]$$

Put this potential in cylindrical coordinates (ρ, θ, z):

$$\Phi(x,y) = \frac{-\lambda}{4\pi\epsilon_0}[\ln\left(\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta-\theta_0)\right) - \ln\left(\rho^2+\rho_0^2+2\rho\rho_0\cos(\theta+\theta_0)\right) \\ -\ln\left(\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta+\theta_0)\right) + \ln\left(\rho^2+\rho_0^2+2\rho\rho_0\cos(\theta-\theta_0)\right)]$$

Divide everything by ρ^2 so that we can get everything in terms of ρ_0/ρ and then we are able to make a statement about being far away from the origin:

$$\Phi(x,y) = \frac{-\lambda}{4\pi\epsilon_0}[\ln(\rho^2) + \ln\left(1+\left(\frac{\rho_0}{\rho}\right)^2 - 2\frac{\rho_0}{\rho}\cos(\theta-\theta_0)\right) - \ln(\rho^2) - \ln\left(1+\left(\frac{\rho_0}{\rho}\right)^2 + 2\frac{\rho_0}{\rho}\cos(\theta+\theta_0)\right) \\ -\ln(\rho^2) - \ln\left(1+\left(\frac{\rho_0}{\rho}\right)^2 - 2\frac{\rho_0}{\rho}\cos(\theta+\theta_0)\right) + \ln(\rho^2) + \ln\left(1+\left(\frac{\rho_0}{\rho}\right)^2 + 2\frac{\rho_0}{\rho}\cos(\theta-\theta_0)\right)]$$

$$\Phi(x,y) = \frac{-\lambda}{4\pi\epsilon_0}[\ln\left(1+\left(\frac{\rho_0}{\rho}\right)^2 - 2\frac{\rho_0}{\rho}\cos(\theta-\theta_0)\right) - \ln\left(1+\left(\frac{\rho_0}{\rho}\right)^2 + 2\frac{\rho_0}{\rho}\cos(\theta+\theta_0)\right) \\ -\ln\left(1+\left(\frac{\rho_0}{\rho}\right)^2 - 2\frac{\rho_0}{\rho}\cos(\theta+\theta_0)\right) + \ln\left(1+\left(\frac{\rho_0}{\rho}\right)^2 + 2\frac{\rho_0}{\rho}\cos(\theta-\theta_0)\right)]$$

Expand each term in a Taylor series using $\ln(1+x) = x - x^2/2 + x^3/3 + \ldots$

$$\Phi(x,y)=\frac{-\lambda}{4\pi\epsilon_0}[\left[\left(\frac{\rho_0}{\rho}\right)^2-2\frac{\rho_0}{\rho}\cos(\theta-\theta_0)\right]-(1/2)\left[\left(\frac{\rho_0}{\rho}\right)^2-2\frac{\rho_0}{\rho}\cos(\theta-\theta_0)\right]^2+F_1(x^3,x^4,\ldots)$$
$$-\left[\left(\frac{\rho_0}{\rho}\right)^2+2\frac{\rho_0}{\rho}\cos(\theta+\theta_0)\right]+(1/2)\left[\left(\frac{\rho_0}{\rho}\right)^2+2\frac{\rho_0}{\rho}\cos(\theta+\theta_0)\right]^2+F_2(x^3,x^4,\ldots)$$
$$-\left[\left(\frac{\rho_0}{\rho}\right)^2-2\frac{\rho_0}{\rho}\cos(\theta+\theta_0)\right]+(1/2)\left[\left(\frac{\rho_0}{\rho}\right)^2-2\frac{\rho_0}{\rho}\cos(\theta+\theta_0)\right]^2+F_3(x^3,x^4,\ldots)$$
$$+\left[\left(\frac{\rho_0}{\rho}\right)^2+2\frac{\rho_0}{\rho}\cos(\theta-\theta_0)\right]-(1/2)\left[\left(\frac{\rho_0}{\rho}\right)^2+2\frac{\rho_0}{\rho}\cos(\theta-\theta_0)\right]^2+F_4(x^3,x^4,\ldots)]$$

Most of the first few terms cancel out when expanded:

$$\Phi(x,y)=\frac{-\lambda}{4\pi\epsilon_0}[4\left(\frac{\rho_0}{\rho}\right)^2(\cos^2(\theta+\theta_0)-\cos^2(\theta-\theta_0))$$
$$+F_1(x^3,x^4,\ldots)+F_2(x^3,x^4,\ldots)+F_3(x^3,x^4,\ldots)+F_4(x^3,x^4,\ldots)]$$

Far away from the origin we have $\rho >> \rho_0$ and therefore $\rho_0/\rho<<1$. This means that $(\rho_0/\rho)^3$ and $(\rho_0/\rho)^4$ etc. are negligible compared to $(\rho_0/\rho)^2$ and they can all be dropped.

$$\Phi(x,y)=\frac{-\lambda}{4\pi\epsilon_0}4\left(\frac{\rho_0}{\rho}\right)^2(\cos^2(\theta+\theta_0)-\cos^2(\theta-\theta_0))$$

$$\Phi(x,y)=\frac{-\lambda}{\pi\epsilon_0}\left(\frac{\rho_0}{\rho}\right)^2(-4\cos\theta\cos\theta_0\sin\theta\sin\theta_0)$$

$$\boxed{\Phi(x,y)=\frac{4\lambda}{\pi\epsilon_0}\frac{(x_0y_0)(xy)}{\rho^4}}$$

This is the quadrupole term, which makes sense because there are four line charges.