Jackson 2.28 Homework Problem Solution

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PROBLEM:

A closed volume is bounded by conducting surfaces that are the n sides of a regular polyhedron (n = 4, 6, 8, 12, 20). The n surfaces are at different potentials V_i, i = 1, 2, …, n. Prove in the simplest way you can that the potential at the center of the polyhedron is the average potential on the n sides. This problem bears on Problem 2.23b, and has an interesting similarity to the result of Problem 1.10.

SOLUTION:

Because the potential obeys the superposition principle, we can break down the total potential at the center point in the polyhedron due to the n sides at different potentials V_i as a sum of potentials, each one due to one side being held at its potential V_i and all other sides being held at zero:

$$\Phi(V_1, V_2, V_3, .., V_n) = \Phi(V_1, 0, 0, .., 0) + \Phi(0, V_2, 0, .., 0) + ... + \Phi(0, 0, 0, .., V_n)$$

$$\Phi(V_1, V_2, V_3, .., V_n) = \sum_{i=1}^{n} \Phi(i^{\text{th}} \text{ side at } V_i, \text{ all others at } 0)$$

$$\Phi(V_1, V_2, V_3, .., V_n) = \sum_{i=1}^{n} \Phi_i \quad \text{where} \quad \Phi_i = \Phi_i(i^{\text{th}} \text{ side at } V_i, \text{ all others at } 0)$$

Now consider a special case where all the walls are held at the same potential $Vi = V$. In this case, the relaxation method tells us that the entire interior, including the center point must also be at this constant potential so that $\Phi = V$. Because the center point is equally distant from each face and they all have the same area, the potentials at the center due to each face are all equal if all the faces are equal: $\Phi_i = \Phi_0$. Plugging these values into the last equation above, for this special case, we have:

$$V = \sum_{i=1}^{n} \Phi_0$$

$$V = n\Phi_0$$

$$\Phi_0 = \frac{V}{n}$$

The contribution at the center from each face is therefore V/n where V is the potential of the face. Now, if a face a held at a potential of zero, it obviously contributes nothing in the sense of the superposition of potentials. (Surfaces at zero potential obviously contribute to a problem overall because they provide boundary conditions that must be met. A surface at zero potential is very different from no surface.) This means that if all other faces are held at a potential of zero except face i which is held at potential V_i, the potential at the center will be V_i/n by the equation above:

$$\Phi_i = \frac{V_i}{n}$$

Note that this is only true at the center where all faces are equally distant from the observation point. Adding up all the potential components (plugging this into the general equation at the beginning), we have:

$$\Phi = \sum_{i=1}^{n} \left(\frac{V_i}{n} \right)$$

$$\boxed{\Phi = \frac{\sum_{i=1}^{n} V_i}{n}}$$

This is just the average over all the potentials of each individual face.

Another way to solve this is in terms of the Green's function method. There is no charge inside the polyhedron, so that the Green's function method solution becomes:

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \oint \left(\Phi \frac{d G_D}{d n'} \right) da'$$

This is just an integral over the entire surface, so we can break it up into an integral over each face:

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \sum_{i=1}^{n} \int_{S_i} \left(V_i \frac{d G_{D,i}}{d n'} \right) da'$$

Now the potential is constant across an entire given face, so that it can come out of the integral:

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \sum_{i=1}^{n} V_i \int_{S_i} \frac{d G_{D,i}}{d n'} da'$$

The integral at this point is entirely geometrical. The Green's function depends entirely on the geometry of the surface and the distance between observation point and the surface. Because each face of a regular polyhedron has the exact same shape, area, angle, and distance from the center, the integral must be the same for all surface and can come out of the summation symbol:

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \int_{S_i} \frac{d G_{D,i}}{d n'} da' \sum_{i=1}^{n} V_i$$

At this point, we can suck everything in front into one constant:

$$\Phi(\mathbf{x}) = C \sum_{i=1}^{n} V_i$$

Similar to as was done before, we can find the constant C by taking the special case of all surfaces at the same potential V leading to points inside being at the potential V:

$$V = C \sum_{i=1}^{n} V$$

$$V = C\, n\, V$$

$$1 = C\, n$$

$$C = \frac{1}{n}$$

So that finally:

$$\boxed{\Phi = \frac{\sum_{i=1}^{n} V_i}{n}}$$