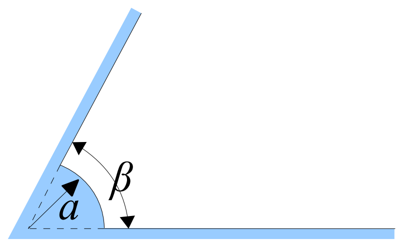
Jackson 2.26 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell

PROBLEM:

The two-dimensional region, $\rho \geq a$, $0 \leq \varphi \leq \beta$, is bounded by conducting surfaces at $\varphi = 0$, $\rho = a$, and $\varphi = \beta$ held at zero potential, as indicated in the sketch. At large ρ the potential is determined by some configuration of charges and/or conductors at fixed potentials.



(a) Write down a solution for the potential $\Phi(\rho, \phi)$ that satisfies the boundary conditions for finite ρ.

(b) Keeping only the lowest non-vanishing terms, calculate the electric field components E_ρ and E_φ and also the surface-charge densities $\sigma(\rho, 0)$, $\sigma(\rho, \beta)$, and $\sigma(a, \varphi)$ on the three boundary surfaces.

(c) Consider $\beta = \pi$ (a plane conductor with a half-cylinder of radius a on it). Show that far from the half-cylinder, the lowest order terms of part b give a uniform electric field normal to the plane. Sketch the charge density on and in the neighborhood of the half-cylinder. For fixed electric field strength far from the plane, show that the total charge on the half-cylinder (actually charge per unit length in the z direction) is twice as large as would reside on a strip of width $2a$ in its absence. Show that the extra portion is drawn from regions of the plane nearby, so that the total charge on a strip of width large compared to a is the same whether the half-cylinder is there or not.

SOLUTION:

We can think of the charges away from this rounded corner as external to the problem, so that they simply create some boundary condition on the potential at large ρ. The region near the corner has no charges and is described by the Laplace equation.

$$\nabla^2 \Phi = 0$$

In polar coordinates, the Laplace equation becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Using the method of separation of variables, the general solution is found to be

$$\Phi(\rho,\phi)=(a_0+b_0\ln\rho)(A_0+B_0\phi)+\sum_{\nu,\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$

Apply the boundary condition $\Phi(\rho,\phi=0)=0$

$$0=(a_0+b_0\ln\rho)(A_0)+\sum_{\nu,\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\left(A_\nu+B_\nu\right)$$

To hold true for all values of ρ we must have $A_0=0$ and $B_\nu=-A_\nu$. The solution now becomes:

$$\Phi(\rho,\phi)=(a_0+b_0\ln\rho)(B_0\phi)+\sum_{\nu,\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)A_\nu\sin(\nu\phi)$$

Apply the boundary condition $\Phi(\rho,\phi=\beta)=0$

$$0=(a_0+b_0\ln\rho)(B_0\beta)+\sum_{\nu,\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)A_\nu\sin(\nu\beta)$$

To hold true for all values of ρ we must have $B_0=0$ and $\nu=\frac{n\pi}{\beta}$ where n = 1, 2, 3... which gives

$$\Phi(\rho,\phi)=\sum_{n=1}^{\infty}\left(a_n\rho^{n\pi/\beta}+b_n\rho^{-n\pi/\beta}\right)A_n\sin\left(\frac{n\pi\phi}{\beta}\right)$$

Apply the boundary condition $\Phi(\rho=a,\phi)=0$

$$0=\sum_{n=1}^{\infty}\left(a_n a^{n\pi/\beta}+b_n a^{-n\pi/\beta}\right)A_n\sin\left(\frac{n\pi\phi}{\beta}\right)$$

To hold true for all values of angles we must have $0=a_n a^{n\pi/\beta}+b_n a^{-n\pi/\beta}$ which leads to:

$$b_n=-a_n a^{2n\pi/\beta}$$

The solution at this point takes the form (where several constant factors have been combined with the last remaining undetermined constant):

$$\boxed{\Phi(\rho,\phi)=\sum_{n=1}^{\infty}A_n\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta}-\left(\frac{\rho}{a}\right)^{-n\pi/\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)}$$

(b) Keeping only the lowest non-vanishing terms, calculate the electric field components E_ρ and E_φ and also the surface-charge densities $\sigma(\rho, 0)$, $\sigma(\rho, \beta)$, and $\sigma(a, \varphi)$ on the three boundary surfaces.

The electric field defined in terms of the electric potential is:

$$\mathbf{E}=-\nabla\Phi$$

In polar coordinates this becomes:

$$\mathbf{E} = -\hat{\boldsymbol{\rho}}\frac{\partial\Phi}{\partial\rho} - \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}$$

or presented differently:

$$E_\rho = -\frac{\partial\Phi}{\partial\rho} \quad \text{and} \quad E_\phi = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}$$

$$E_\rho = -\frac{\partial}{\partial\rho}\sum_{n=1}^{\infty} A_n\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)$$

$$E_\rho = -\sum_{n=1}^{\infty} A_n\frac{n\pi}{a\beta}\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta-1} + \left(\frac{\rho}{a}\right)^{-n\pi/\beta-1}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)$$

The lowest non-vanishing term is:

$$\boxed{E_\rho = -A_1\frac{\pi}{a\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} + \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)\sin\left(\frac{\pi\phi}{\beta}\right)}$$

The other component can also be easily done:

$$E_\phi = -\frac{1}{\rho}\frac{\partial}{\partial\phi}\sum_{n=1}^{\infty} A_n\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)$$

$$E_\phi = -\sum_{n=1}^{\infty} A_n\frac{n\pi}{a\beta}\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta-1}\right)\cos\left(\frac{n\pi\phi}{\beta}\right)$$

The lowest non-vanishing term is:

$$\boxed{E_\phi = -A_1\frac{\pi}{a\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)\cos\left(\frac{\pi\phi}{\beta}\right)}$$

The total electric field with all the terms is then:

$$\mathbf{E} = \hat{\boldsymbol{\rho}}E_\rho + \hat{\boldsymbol{\phi}}E_\phi$$

$$\mathbf{E} = \sum_{n=1}^{\infty} A_n\frac{n\pi}{a\beta}\left[-\hat{\boldsymbol{\rho}}\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta-1} + \left(\frac{\rho}{a}\right)^{-n\pi/\beta-1}\right)\sin\left(\frac{n\pi\phi}{\beta}\right) - \hat{\boldsymbol{\phi}}\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta-1}\right)\cos\left(\frac{n\pi\phi}{\beta}\right)\right]$$

The surface charge density $\sigma(\rho, 0)$ is found using the pillbox Gaussian surface, which yields:

$$\sigma(\rho, 0) = \left[\epsilon_0\mathbf{E}\cdot\mathbf{n}\right]_{n=n_0}$$

$$\sigma(\rho,0) = \left[\epsilon_0 \mathbf{E}\cdot\hat{\boldsymbol{\phi}}\right]_{\phi=0}$$

$$\sigma(\rho,0) = \left[\epsilon_0 E_\phi\right]_{\phi=0}$$

$$\sigma(\rho,0) = \left[-A_1 \frac{\pi\,\epsilon_0}{a\,\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)\cos\left(\frac{\pi\,\phi}{\beta}\right)\right]_{\phi=0}$$

$$\boxed{\sigma(\rho,0) = -A_1 \frac{\pi\,\epsilon_0}{a\,\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)}$$

The surface charge density $\sigma(\rho, \beta)$ is found in a similar manner:

$$\sigma(\rho,\beta) = \left[\epsilon_0 \mathbf{E}\cdot(-\hat{\boldsymbol{\phi}})\right]_{\phi=\beta}$$

$$\sigma(\rho,\beta) = \left[-\epsilon_0 E_\phi\right]_{\phi=\beta}$$

$$\sigma(\rho,\beta) = \left[A_1 \frac{\pi\,\epsilon_0}{a\,\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)\cos\left(\frac{\pi\,\phi}{\beta}\right)\right]_{\phi=\beta}$$

$$\boxed{\sigma(\rho,\beta) = -A_1 \frac{\pi\,\epsilon_0}{a\,\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)}$$

The charge densities on the two flat surfaces are equal, which is what we would expect because of the symmetry of the problem.

The surface charge density $\sigma(a, \varphi)$ obeys:

$$\sigma(a,\phi) = \left[\epsilon_0 \mathbf{E}\cdot\hat{\boldsymbol{\rho}}\right]_{\rho=a}$$

$$\sigma(a,\phi) = \left[\epsilon_0 E_\rho\right]_{\rho=a}$$

$$\sigma(a,\phi) = \left[-A_1 \frac{\pi\,\epsilon_0}{a\,\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} + \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)\sin\left(\frac{\pi\,\phi}{\beta}\right)\right]_{\rho=a}$$

$$\boxed{\sigma(a,\phi) = -A_1 \frac{2\,\pi\,\epsilon_0}{a\,\beta}\sin\left(\frac{\pi\,\phi}{\beta}\right)}$$

(c) Consider $\beta = \pi$ (a plane conductor with a half-cylinder of radius a on it). Show that far from the half-cylinder, the lowest order terms of part b give a uniform electric field normal to the plane. Sketch the charge density on and in the neighborhood of the half-cylinder. For fixed electric field strength far from the plane, show that the total charge on the half-cylinder (actually charge per unit length in the z direction) is twice as large as would reside on a strip of width $2a$ in its absence. Show that the extra portion is drawn from regions of the plane nearby, so that the total charge on a strip of width large compared to a is the same whether the half-cylinder is there or not.

If $\beta = \pi$, the solution for the total electric field reduces to:

$$\mathbf{E}=\sum_{n=1}^{\infty} A_n \frac{n}{a}\left[-\hat{\boldsymbol{\rho}}\left(\left(\frac{\rho}{a}\right)^{n-1}+\left(\frac{\rho}{a}\right)^{-n-1}\right)\sin(n\phi)-\hat{\boldsymbol{\phi}}\left(\left(\frac{\rho}{a}\right)^{n-1}-\left(\frac{\rho}{a}\right)^{-n-1}\right)\cos(n\phi)\right]$$

the lowest order term is:

$$\mathbf{E}=A_1\frac{1}{a}\left[-\hat{\boldsymbol{\rho}}\left(1+\left(\frac{a}{\rho}\right)^2\right)\sin(\phi)-\hat{\boldsymbol{\phi}}\left(1-\left(\frac{a}{\rho}\right)^2\right)\cos(\phi)\right]$$

Far away from the half-cylinder, $\rho >> a$, and thus $1 >> (a/\rho)$ and $1 >> (a/\rho)^2$ so that

$$\mathbf{E}=-A_1\frac{1}{a}\left[\hat{\boldsymbol{\rho}}\sin(\phi)+\hat{\boldsymbol{\phi}}\cos(\phi)\right]$$

$$\boxed{\mathbf{E}=-\frac{A_1}{a}\hat{\mathbf{j}}}$$

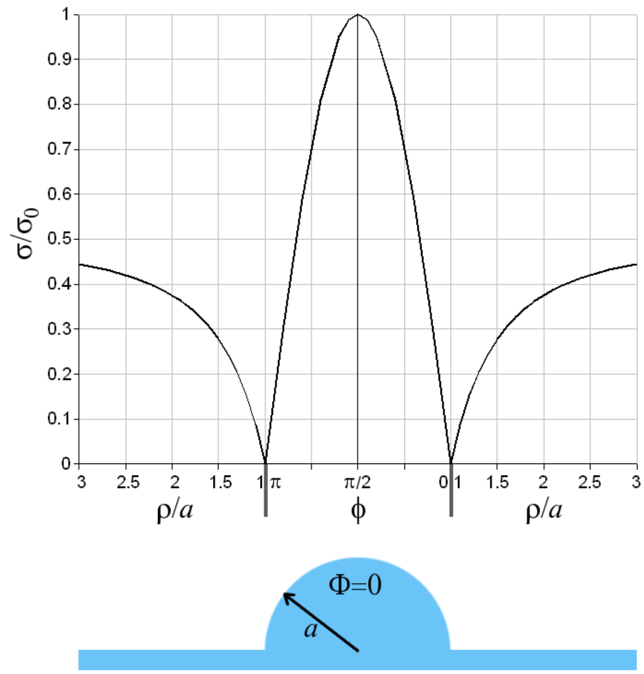
This is just a uniform electric field in the y direction, normal to the conducting plane.

The charge density on the half cylinder has the form

$$\boxed{\sigma(a,\phi)=\sigma_0\sin(\phi)} \quad \text{where} \quad \sigma_0=-A_1\frac{2\epsilon_0}{a}$$

and on the sides:

$$\boxed{\sigma(\rho,0)=\sigma(\rho,\beta)=\frac{\sigma_0}{2}\left(1-\left(\frac{\rho}{a}\right)^{-2}\right)}$$



The total charge on the half-cylinder (per unit length in the z direction) is

$$Q_{\text{half-cyl}} = -A_1 \frac{2\epsilon_0}{a} \int_0^{\pi} \sin(\phi)\, a\, d\phi$$

$$Q_{\text{half-cyl}} = -A_1 4 \epsilon_0$$

If the same electric field were used ($\mathbf{E} = -A_1 / a\, \hat{\mathbf{j}}$ as derived above) and the cylinder were replaced with a strip $2a$ wide, the charge density on the strip would be

$$\sigma(y=0) = \left[\epsilon_0 \mathbf{E} \cdot \mathbf{j}\right]_{y=0}$$

$$\sigma(y=0) = -\epsilon_0 A_1 / a$$

The total charge (per unit length in the z direction) on the strip is:

$$Q_{\text{strip}} = \sigma\, 2a$$

$$Q_{\text{strip}} = -A_1 2 \epsilon_0$$

So that $\boxed{Q_{\text{strip}} = \frac{1}{2} Q_{\text{half-cyl}}}$

Consider a larger strip with width l that includes the central region already focused on. The total charge with the half-cylinder included is:

$$Q_1 = 2\int_a^l (-A_1)\frac{\epsilon_0}{a}\left(1-\left(\frac{\rho}{a}\right)^{-2}\right)d\,\rho + Q_{\text{half-cyl}}$$

$$Q_1 = -2(A_1)\frac{\epsilon_0}{a}(l-a) - 2(A_1)a\,\epsilon_0(1/l - 1/a) + Q_{\text{half-cyl}}$$

$$Q_1 = 2A_1\epsilon_0\left[\frac{-l}{a} - \frac{a}{l}\right]$$

For $l >> a$:

$$Q_1 = \frac{-2l\,\epsilon_0 A_1}{a}$$

The total charge without the cylinder is

$$Q_2 = \sigma\,2l$$

$$Q_2 = \left[\epsilon_0 E_y\right]_{y=0}\sigma\,2l$$

$$Q_2 = (-\epsilon_0 A_1/a)\,2l$$

$$Q_2 = \frac{-2l\,\epsilon_0 A_1}{a}$$

It now becomes apparent that $Q_1 = Q_2$. The total charge over a large region is the same with and without the half-cylinder. This means that the extra charge on the half-cylinder is drawn from the regions nearby on the planes.