Jackson 2.25 Homework Problem Solution

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PROBLEM:

Two conducting planes at zero potential meet along the *z* axis, making an angle β between them, as in Fig. 2.12. A unit line charge parallel to the *z* axis is located between the planes at position (ρ', φ').

(a) Show that ($4\pi\varepsilon_0$) times the potential in the space between the planes, that is, the Dirichlet Green function $G(\rho, \varphi; \rho', \varphi')$, is given by the infinite series

$$G(\rho,\phi,\rho',\phi') = 4\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)$$

(b) By means of complex-variable techniques or other means, show that the series can be summed to give a closed form,

$$G(\rho,\phi,\rho',\phi') = \ln\left[\frac{(\rho)^{2\pi/\beta}+(\rho')^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos[\pi(\phi+\phi')/\beta]}{(\rho)^{2\pi/\beta}+(\rho')^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos[\pi(\phi-\phi')/\beta]}\right]$$

(c) Verify that you obtain the familiar results when $\beta = \pi$ and $\beta = \pi/2$.

SOLUTION:

(a) Split the space between the planes into two regions, region I where $\rho < \rho'$ and region II where $\rho > \rho'$. Each region is charge free, so we can solve the Laplace equation in two-dimensions in polar coordinates. The line charge will come into play when we apply boundary conditions linking the two regions.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

Try a solution of the form $\Phi(\rho,\phi)=R(\rho)\Psi(\phi)$ to find

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)=-\frac{1}{\Psi(\phi)}\frac{\partial^2\Psi(\phi)}{\partial\phi^2}$$

Both sides are now independent and can be set to a constant.

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)=\nu^2 R(\rho) \quad\text{and}\quad \frac{\partial^2\Psi(\phi)}{\partial\phi^2}=-\nu^2\Psi(\phi)$$

The general solution is:

$$\Phi(\rho,\phi)=(a_0+b_0\ln\rho)(A_0+B_0\phi)+\sum_{\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$

Apply the boundary condition at the bottom face:

$$\Phi(\rho,\phi=0)=0$$

$$0=(a_0+b_0\ln\rho)(A_0)+\sum_{\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)(A_\nu+B_\nu)$$

$$A_0=0 \quad \text{and} \quad B_\nu=-A_\nu$$

The solution becomes:

$$\Phi(\rho,\phi)=(a_0+b_0\ln\rho)\phi+\sum_{\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\sin(\nu\phi)$$

Apply the boundary condition at the top face:

$$\Phi(\rho,\phi=\beta)=0$$

$$0=(a_0+b_0\ln\rho)\beta+\sum_{\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\sin(\nu\beta)$$

$a_0=0$ and $b_0=0$ and $0=\sin(\nu\beta)$ which leads to $\nu=\dfrac{m\pi}{\beta}$ where m = 1, 2...

The solution becomes:

$$\Phi(\rho,\phi)=\sum_m\left(a_m\rho^{m\pi/\beta}+b_m\rho^{-m\pi/\beta}\right)\sin(m\pi\phi/\beta) \quad \text{where } m = 1, 2...$$

The solution in both regions must have this form. We must now look at each region separately to get any farther. In the region close to the origin (I), we must have a valid solution at the origin, so that $b_m = 0$, leading to:

$$\Phi_{\mathrm{I}}(\rho,\phi)=\sum_m a_m\rho^{m\pi/\beta}\sin(m\pi\phi/\beta)$$

In the region far from the origin (II), we must have a valid solution at infinity, so that $a_m = 0$, leading to:

$$\Phi_{\mathrm{II}}(\rho,\phi)=\sum_m b_m\rho^{-m\pi/\beta}\sin(m\pi\phi/\beta)$$

We now apply boundary conditions to link the two regions, remembering that there is a line charge present at the boundary:

$$(\mathbf{E}_2-\mathbf{E}_1)\cdot\hat{\mathbf{n}}=\frac{\sigma}{\epsilon_0}$$

$$\left[-\frac{\partial\Phi_{II}}{\partial\rho}+\frac{\partial\Phi_{I}}{\partial\rho}\right]_{\text{at }\rho=\rho'}=\frac{\lambda\,\delta(\phi-\phi')}{\rho'\epsilon_0}$$

$$\sum_m m\sin(m\pi\phi/\beta)[b_m\rho'^{-m\pi/\beta}+a_m\rho'^{m\pi/\beta}]=\frac{\beta}{\pi}\frac{\lambda\,\delta(\phi-\phi')}{\epsilon_0}$$

Multiply both side by a sine and integrate with respect to the polar angle:

$$\sum_m m\int_0^\beta \sin(m\pi\phi/\beta)\sin(n\pi\phi/\beta)\,d\phi\,[b_m\rho'^{-m\pi/\beta}+a_m\rho'^{m\pi/\beta}]=\frac{\beta}{\pi}\frac{\lambda}{\epsilon_0}\sin(n\pi\phi'/\beta)$$

$$\boxed{[b_m\rho'^{-m\pi/\beta}+a_m\rho'^{m\pi/\beta}]=\frac{2}{\pi}\frac{\lambda}{m\epsilon_0}\sin(m\pi\phi'/\beta)}$$

The other boundary condition is:

$$\mathbf{n}\times\mathbf{E}_2=\mathbf{n}\times\mathbf{E}_1$$

$$\left[\frac{\partial\Phi_{I}}{\partial\phi}=\frac{\partial\Phi_{II}}{\partial\phi}\right]_{\text{at }\rho=\rho'}$$

$$\boxed{a_m\rho'^{m\pi/\beta}=b_m\rho'^{-m\pi/\beta}}$$

Solving the system of equations in two independent variables as represented by the above equations in boxes, we find:

$$\boxed{b_m=\rho'^{m\pi/\beta}\frac{1}{\pi}\frac{\lambda}{m\epsilon_0}\sin(m\pi\phi'/\beta)}\qquad\boxed{a_m=\rho'^{-m\pi/\beta}\frac{1}{\pi}\frac{\lambda}{m\epsilon_0}\sin(m\pi\phi'/\beta)}$$

The final solution becomes:

$$\boxed{\Phi(\rho,\phi)=\frac{1}{\pi}\frac{\lambda}{\epsilon_0}\sum_m\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)}$$

If we set $\lambda=4\pi\epsilon_0$, this potential becomes the Green function:

$$\boxed{G(\rho,\phi,\rho',\phi')=4\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)}$$

(b) Let us try to convert the sum into closed form. First note that the sine function is the imaginary part of the complex exponential, so that the Green function becomes:

$$G(\rho,\phi,\rho',\phi')=4\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}\Im\left(e^{im\pi\phi/\beta}\right)\Im\left(e^{im\pi\phi'/\beta}\right)$$

Next use the identity: $2\Im(z_1)\Im(z_2)=\Re\left[-z_1z_2+z_1^*z_2\right]$

$$G(\rho,\phi,\rho',\phi')=\Re\left[-2\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}e^{im\pi(\phi+\phi')/\beta}+2\sum_{m=1}^{\infty}\frac{1}{m}\rho_<^{m\pi/\beta}\rho_>^{-m\pi/\beta}e^{im\pi(\phi-\phi')/\beta}\right]$$

$$G(\rho,\phi,\rho',\phi')=\Re\left[-2\sum_{m=1}^{\infty}\frac{1}{m}Z_1^m+2\sum_{m=1}^{\infty}\frac{1}{m}Z_2^m\right]$$

where $Z_1=\rho_<^{\pi/\beta}\rho_>^{-\pi/\beta}e^{i\pi(\phi+\phi')/\beta}$ and $Z_2=\rho_<^{\pi/\beta}\rho_>^{-\pi/\beta}e^{i\pi(\phi-\phi')/\beta}$

Now use $\sum_{m=1}^{\infty}\frac{Z^m}{m}=-\ln(1-Z)$

$$G(\rho,\phi,\rho',\phi')=2\Re\left[\ln(1-Z_1)-\ln(1-Z_2)\right]$$

$$G(\rho,\phi,\rho',\phi')=2\Re\left[\ln\left(\frac{1-Z_1}{1-Z_2}\right)\right]$$

Use $2\Re[\ln z]=\ln(zz^*)$:

$$G(\rho,\phi,\rho',\phi')=\ln\left(\frac{1+|Z_1|^2-2\Re(Z_1)}{1+|Z_2|^2-2\Re(Z_2)}\right)$$

Insert back in Z_1 and Z_2:

$$G(\rho,\phi,\rho',\phi')=\ln\left(\frac{1+\rho_<^{2\pi/\beta}\rho_>^{-2\pi/\beta}-2\rho_<^{\pi/\beta}\rho_>^{-\pi/\beta}\cos(\pi(\phi+\phi')/\beta)}{1+\rho_<^{2\pi/\beta}\rho_>^{-2\pi/\beta}-2\rho_<^{\pi/\beta}\rho_>^{-\pi/\beta}\cos(\pi(\phi-\phi')/\beta)}\right)$$

$$G(\rho,\phi,\rho',\phi')=\ln\left(\frac{\rho_>^{2\pi/\beta}+\rho_<^{2\pi/\beta}-2(\rho_<\rho_>)^{\pi/\beta}\cos(\pi(\phi+\phi')/\beta)}{\rho_>^{2\pi/\beta}+\rho_<^{2\pi/\beta}-2(\rho_<\rho_>)^{\pi/\beta}\cos(\pi(\phi-\phi')/\beta)}\right)$$

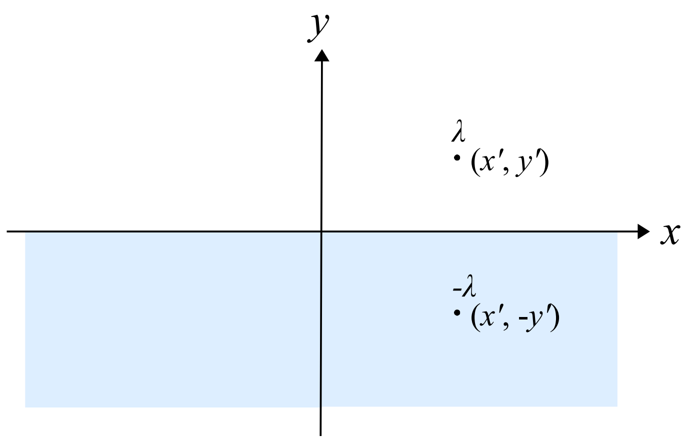
Because of the symmetry, we can drop the greater-than and less-than subscripts:

$$\boxed{G(\rho,\phi,\rho',\phi')=\ln\left(\frac{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos(\pi(\phi+\phi')/\beta)}{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos(\pi(\phi-\phi')/\beta)}\right)}$$

(c) For $\beta = \pi$ this reduces to:

$$\boxed{\frac{1}{4\pi\epsilon_0} G(\rho, \phi, \rho', \phi') = \frac{1}{2\pi\epsilon_0} \ln(|\mathbf{x} - \mathbf{x}''|) - \frac{1}{2\pi\epsilon_0} \ln(|\mathbf{x} - \mathbf{x}'|)}$$

This is the potential of a unit line charge located at **x**' near a grounded conducting plane in the *x*-*z* axis, which is effectively the same as a negative image line charge at the mirror location **x**''.



For $\beta = \pi/2$ this reduces to:

$$G(\rho, \phi, \rho', \phi') = \ln\left(\frac{\rho^4 + \rho'^4 - 2(\rho\rho')^2 \cos(2(\phi+\phi'))}{\rho^4 + \rho'^4 - 2(\rho\rho')^2 \cos(2(\phi-\phi'))}\right)$$

$$\boxed{\begin{aligned}\frac{1}{4\pi\epsilon_0} G(\rho, \phi, \rho', \phi') = -\frac{1}{2\pi\epsilon_0}[&\ln\left(\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')}\right) - \ln\left(\sqrt{\rho^2 + \rho'^2 + 2\rho\rho' \cos(\phi+\phi')}\right) \\ &- \ln\left(\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi+\phi')}\right) + \ln\left(\sqrt{\rho^2 + \rho'^2 + 2\rho\rho' \cos(\phi-\phi')}\right)]\end{aligned}}$$

This is the potential of a unit line charge located at **x**' near a right-angle interior edge, which is equivalent to three image charges, two negative ones offset by plus and minus 90 degrees, and one positive one offset by 180 degrees. This result matches the one found in Problem 2.3.