Jackson 2.23 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell

C. S. BAIRD

UMASS LOWELL

PROBLEM:

A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at constant potential V. The other four sides are at zero potential.

(a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

(b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28.

(c) Find the surface-charge density on the surface $z = a$.

SOLUTION:

The problem contains no charge, so the electric potential is described everywhere inside the cube by the Laplace equation:

$$\nabla^2 \Phi = 0$$

This problem has a boundary, the cube, that best matches rectangular coordinates. In rectangular coordinates, the Laplace equation becomes:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Using the method of separation of variables, the most general solution to this equation is:

$$\begin{aligned}\Phi(x, y, z) = &(A_{\alpha 0} + B_{\alpha 0} x)(A_{\beta 0} + B_{\beta 0} y)(A_{\gamma 0} + B_{\gamma 0} z)\\
&+ \sum_{\beta} (A_{\alpha 0} + B_{\alpha 0} x)(A_{\beta} e^{i\beta y} + B_{\beta} e^{-i\beta y})(A_{\gamma} e^{\gamma z} + B_{\gamma} e^{-\gamma z})\\
&+ \sum_{\alpha} (A_{\alpha} e^{i\alpha x} + B_{\alpha} e^{-i\alpha x})(A_{\beta 0} + B_{\beta 0} y)(A_{\gamma} e^{\gamma z} + B_{\gamma} e^{-\gamma z})\\
&+ \sum_{\alpha, \beta} (A_{\alpha} e^{i\alpha x} + B_{\alpha} e^{-i\alpha x})(A_{\beta} e^{i\beta y} + B_{\beta} e^{-i\beta y})(A_{\gamma} e^{\gamma z} + B_{\gamma} e^{-\gamma z})\end{aligned}$$

Apply the boundary condition, $\Phi(x = 0, y, z) = 0$

$$
\begin{aligned}
0=&(A_{\alpha 0})(A_{\beta 0}+B_{\beta 0}\,y)(A_{\gamma 0}+B_{\gamma 0}\,z)\\
&+\sum_{\beta}(A_{\alpha 0})(A_{\beta}e^{i\beta y}+B_{\beta}e^{-i\beta y})(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})\\
&+\sum_{\alpha}(A_{\alpha}+B_{\alpha})(A_{\beta 0}+B_{\beta 0}\,y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})\\
&+\sum_{\alpha,\beta}(A_{\alpha}+B_{\alpha})(A_{\beta}e^{i\beta y}+B_{\beta}e^{-i\beta y})(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})
\end{aligned}
$$

This must be true for all y and z, so that each term must vanish separately.
This forces $A_{\alpha 0} = 0$ and $B_{\alpha} = -A_{\alpha}$. The solution now becomes:

$$
\begin{aligned}
\Phi(x,y,z)=&(B_{\alpha 0}\,x)(A_{\beta 0}+B_{\beta 0}\,y)(A_{\gamma 0}+B_{\gamma 0}\,z)\\
&+\sum_{\beta}(B_{\alpha 0}\,x)(A_{\beta}e^{i\beta y}+B_{\beta}e^{-i\beta y})(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})\\
&+\sum_{\alpha}A_{\alpha}\sin(\alpha x)(A_{\beta 0}+B_{\beta 0}\,y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})\\
&+\sum_{\alpha,\beta}A_{\alpha}\sin(\alpha x)(A_{\beta}e^{i\beta y}+B_{\beta}e^{-i\beta y})(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})
\end{aligned}
$$

Similarly, the boundary condition, $\Phi(x, y = 0, z) = 0$ leads to $A_{\beta 0} = 0$ and $B_{\beta} = -A_{\beta}$ which gives the solution:

$$
\begin{aligned}
\Phi(x,y,z)=&(B_{\alpha 0}\,x)(B_{\beta 0}\,y)(A_{\gamma 0}+B_{\gamma 0}z)\\
&+\sum_{\beta}(B_{\alpha 0}\,x)A_{\beta}\sin(\beta y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})\\
&+\sum_{\alpha}A_{\alpha}\sin(\alpha x)(B_{\beta 0}\,y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})\\
&+\sum_{\alpha,\beta}A_{\alpha}\sin(\alpha x)A_{\beta}\sin(\beta y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})
\end{aligned}
$$

Now apply the boundary condition $\Phi(x = a, y, z) = 0$

$$
\begin{aligned}
0=&(B_{\alpha 0}\,a)(B_{\beta 0}\,y)(A_{\gamma 0}+B_{\gamma 0}\,z)\\
&+\sum_{\beta}(B_{\alpha 0}\,a)A_{\beta}\sin(\beta y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})\\
&+\sum_{\alpha}A_{\alpha}\sin(\alpha a)(B_{\beta 0}\,y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})\\
&+\sum_{\alpha,\beta}A_{\alpha}\sin(\alpha a)A_{\beta}\sin(\beta y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})
\end{aligned}
$$

The only way this can be true for all y and z is if $B_{\alpha 0} = 0$ and $\alpha = n\pi/a$ where $n = 0, 1, 2,...$

$$
\Phi(x,y,z)=\sum_{n}A_{n}\sin\left(\frac{n\pi x}{a}\right)(B_{\beta 0}\,y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})+\sum_{n,\beta}A_{n}\sin\left(\frac{n\pi x}{a}\right)A_{\beta}\sin(\beta y)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})
$$

Similarly, apply the boundary condition $\Phi(x, y = a, z) = 0$ to get $B_{\beta 0} = 0$ and $\beta = m\pi/a$ where
$n = 0, 1, 2,...$

$$
\Phi(x,y,z)=\sum_{n,m}A_{n}\sin\left(\frac{n\pi x}{a}\right)A_{m}\sin\left(\frac{m\pi y}{a}\right)(A_{\gamma}e^{\gamma z}+B_{\gamma}e^{-\gamma z})
$$

By definition we have $\gamma^2=\alpha^2+\beta^2$ so that we now know $\gamma=\sqrt{(n^2+m^2)\pi^2/a^2}$. We can combine several constants so that we now have:

$$\Phi(x,y,z)=\sum_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left(A_{n,m}e^{z\sqrt{(n^2+m^2)\pi^2/a^2}}+B_{n,m}e^{-z\sqrt{(n^2+m^2)\pi^2/a^2}}\right)$$

Now apply the boundary condition $\Phi(x, y, z = 0) = V$

$$V=\sum_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)(A_{n,m}+B_{n,m})$$

Multiply both side by $\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)$ and integrate over x and y from 0 to a

$$\int_0^a\int_0^a V\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)dx\,dy$$
$$=\sum_{n,m}\int_0^a\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right)dx\int_0^a\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)dy\,(A_{n,m}+B_{n,m})$$

Due to orthogonality, each integral on the right is zero, except when $n = n'$ and $m = m'$

$$\int_0^a\int_0^a V\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dx\,dy=\frac{a^2}{4}(A_{n,m}+B_{n,m})$$

$$(A_{n,m}+B_{n,m})=\frac{4V}{a^2}\int_0^a\sin\left(\frac{n\pi x}{a}\right)dx\int_0^a\sin\left(\frac{m\pi y}{a}\right)dy$$

$$(A_{n,m}+B_{n,m})=\frac{4V}{a^2}\frac{a}{n\pi}\left[1-(-1)^n\right]\frac{a}{m\pi}\left[1-(-1)^m\right]$$

$$\boxed{A_{n,m}+B_{n,m}=\frac{16V}{nm\pi^2}}\qquad n,\,m=\text{odd}$$

Now apply the final boundary condition $\Phi(x, y, z = \mathrm{a}) = V$

$$V=\sum_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left(A_{n,m}e^{a\sqrt{(n^2+m^2)\pi^2/a^2}}+B_{n,m}e^{-a\sqrt{(n^2+m^2)\pi^2/a^2}}\right)$$

Repeat the process done above to get

$$\boxed{A_{n,m}e^{\sqrt{(n^2+m^2)\pi^2}}+B_{n,m}e^{-\sqrt{(n^2+m^2)\pi^2}}=\frac{16V}{nm\pi^2}}\qquad n,\,m=\text{odd}$$

Solve the system of equations in the boxes above to find:

$$A_{n,m}=\frac{8V}{nm\pi^2}\frac{e^{-\frac{1}{2}\sqrt{(n^2+m^2)\pi^2}}}{\cosh\left((\pi/2)\sqrt{n^2+m^2}\right)}$$

$$B_{n,m}=\frac{8V}{nm\pi^2}\frac{e^{\frac{1}{2}\sqrt{(n^2+m^2)\pi^2}}}{\cosh\left((\pi/2)\sqrt{n^2+m^2}\right)}$$

The final solution is now:

$$\Phi(x,y,z)=\sum_{n,m\text{ odd}}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\frac{8V}{nm\pi^2}\left(\frac{e^{-\frac{1}{2}\sqrt{(n^2+m^2)\pi^2}}e^{z\sqrt{(n^2+m^2)\pi^2/a^2}}+e^{\frac{1}{2}\sqrt{(n^2+m^2)\pi^2}}e^{-z\sqrt{(n^2+m^2)\pi^2/a^2}}}{\cosh\left((\pi/2)\sqrt{n^2+m^2}\right)}\right)$$

$$\boxed{\Phi(x,y,z)=\sum_{n,m\text{ odd}}\frac{16V}{nm\pi^2}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left(\frac{\cosh\left((\pi/2)\sqrt{n^2+m^2}((2z/a)-1)\right)}{\cosh\left((\pi/2)\sqrt{n^2+m^2}\right)}\right)}$$

(b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28.

The potential at the center of the cube is:

$$\Phi(a/2,a/2,a/2)=\frac{16V}{\pi^2}\sum_{n,m\text{ odd}}\frac{1}{nm}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\left(\frac{1}{\cosh\left((\pi/2)\sqrt{n^2+m^2}\right)}\right)$$

$$\begin{aligned}\Phi(a/2,a/2,a/2)=V[&\frac{16}{\pi^2}\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\left(\frac{1}{\cosh\left((\pi/2)\sqrt{2}\right)}\right)\\&+\frac{16}{\pi^2}\frac{1}{3}\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)\left(\frac{1}{\cosh\left((\pi/2)\sqrt{10}\right)}\right)\\&+\frac{16}{\pi^2}\frac{1}{3}\sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\left(\frac{1}{\cosh\left((\pi/2)\sqrt{10}\right)}\right)\\&+\frac{16}{\pi^2}\frac{1}{5}\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{5\pi}{2}\right)\left(\frac{1}{\cosh\left((\pi/2)\sqrt{26}\right)}\right)+\ldots]\end{aligned}$$

$$\Phi(a/2,a/2,a/2)=V[0.3476-0.0075-0.0075+0.0002+\ldots]$$

Here is solution, to three significant figures, if you only keep the first term, or the first two terms, etc:

$$\Phi_{1\text{ term}}(a/2,a/2,a/2)=V[0.348]$$
$$\Phi_{2\text{ terms}}(a/2,a/2,a/2)=V[0.340]$$
$$\Phi_{3\text{ terms}}(a/2,a/2,a/2)=V[0.333]$$
$$\Phi_{4\text{ terms}}(a/2,a/2,a/2)=V[0.333]$$

We only have to keep the first ***three*** terms to have the answer accurate to three significant figures. It is obvious that the solution is converging to:

$$\boxed{\Phi(a/2, a/2, a/2) = \frac{V}{3}}$$

There are six walls on the cube and two sides have a non-zero potential *V*, so the average value of the potential on the sides of the cube is $\Phi_{\text{ave on surf}} = \frac{2V}{6} = \frac{V}{3}$. This leads to the interesting conclusion that:

$$\boxed{\Phi(a/2, a/2, a/2) = \Phi_{\text{ave on surf}}}$$

(c) Find the surface-charge density on the surface $z = a$.

$$\sigma = \left[-\epsilon_0 \frac{d\Phi}{dn}\right]_{n=a}$$

We have solved the potential on the inside of the cube, so we can only use that potential to find the surface charge density on the inside of the $z = a$ surface. The normal to the inside surface is in the negative z direction so that:

$$\sigma = \left[\epsilon_0 \frac{d\Phi}{dz}\right]_{z=a}$$

$$\sigma = \left[\epsilon_0 \frac{d}{dz} \sum_{n,m \text{ odd}} \frac{16V}{nm\pi^2} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \left(\frac{\cosh\left((\pi/2)\sqrt{n^2+m^2}((2z/a)-1)\right)}{\cosh\left((\pi/2)\sqrt{n^2+m^2}\right)}\right)\right]_{z=a}$$

$$\sigma = \left[\epsilon_0 \sum_{n,m \text{ odd}} \frac{16V}{nm\pi^2} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \left(\frac{\sinh\left((\pi/2)\sqrt{n^2+m^2}((2z/a)-1)\right)}{\cosh\left((\pi/2)\sqrt{n^2+m^2}\right)}\right)(\pi/a)\sqrt{n^2+m^2}\right]_{z=a}$$

$$\boxed{\sigma = \frac{16\epsilon_0 V}{\pi a} \sum_{n,m \text{ odd}} \frac{\sqrt{n^2+m^2}}{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \left(\tanh\left((\pi/2)\sqrt{n^2+m^2}\right)\right)}$$