Jackson 2.1 Homework Problem Solution

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PROBLEM:

A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Using the method of images, find:
(a) the surface-charge density induced on the plane, and plot it;
(b) the force between the plane and the charge using Coulomb's law for the force between the charge and its image;
(c) the total force acting on the plane by integrating $\sigma^2/2\epsilon_0$ over the whole plane;
(d) the work necessary to remove the charge q from its position to infinity
(e) the potential energy between the charge q and its image [compare the answer to part d and discuss].
(f) Find the answer to part d in electron volts for an electron originally one angstrom from the surface.

SOLUTION:

We place the point charge q at $z = d$ and its image charge $-q$ at $z = -d$. The total potential is then just the potential due to these two point charges:

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

a) the surface-charge density can be found using the relation we derived in a previous homework problem:

$$\left[\sigma = -\epsilon_0 \frac{\partial\Phi(\mathbf{x})}{\partial n}\right]_{\text{on S}}$$

$$\left[\sigma = -\epsilon_0 \frac{\partial\Phi(\mathbf{x})}{\partial z}\right]_{z=0}$$

$$\left[\sigma = -\epsilon_0 \frac{q}{4\pi\epsilon_0}\left[\frac{-(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right]\right]_{z=0}$$

$$\boxed{\sigma = -\frac{1}{2\pi}\frac{q\,d}{(x^2+y^2+d^2)^{3/2}}}$$

(b) the force on the particle due to the image charge is:

$$\boxed{\mathbf{F} = -\frac{1}{16\pi\epsilon_0}\frac{q^2}{d^2}\hat{\mathbf{z}}}$$

The real charge is attracted down towards the conductor. The force gets stronger as it gets closer.

(c) the force on the conductor should be equal and opposite to the force on the particle, which we derived in part b. We are supposed to calculate it anyways. Let us calculate the force as the interaction of the surface charge and the particle, as opposed to the interaction of the particle with its image.

$$\frac{d\mathbf{F}}{da} = \sigma\mathbf{E}$$

The incremental force per unit area shown on the left is the electrostatic pressure. But the field is also related to the surface charge (when neglecting the field part that would give rise to a self-force) according to:

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

so that

$$\frac{d\mathbf{F}}{da} = \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}}$$ *Electrostatic pressure on the surface of a conductor*

The total force is just the pressure times an incremental patch of area, integrated over all area patches:

$$d\mathbf{F} = \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}}\,da$$

$$\mathbf{F} = \int \frac{\sigma^2}{2\epsilon_0}da\,\hat{\mathbf{n}}$$

$$\mathbf{F} = \frac{1}{2\epsilon_0}\frac{q^2 d^2}{2\pi}\int_0^\infty \frac{\rho\, d\rho}{(\rho^2+d^2)^3}\hat{\mathbf{z}}$$

$$\mathbf{F} = -\hat{\mathbf{z}}\frac{1}{2\epsilon_0}\frac{q^2 d^2}{8\pi}\left[\frac{1}{(\rho^2+d^2)^2}\right]_0^\infty$$

$$\boxed{\mathbf{F} = \frac{1}{16\pi\epsilon_0}\frac{q^2}{d^2}\hat{\mathbf{z}}}$$

d) The work needed to remove the charge to infinity is:

$$W = \int_d^\infty F(l)\,dl$$

$$W = \frac{q^2}{16\pi\epsilon_0}\int_d^\infty \frac{1}{l^2}dl$$

$$W = \frac{q^2}{16\pi\epsilon_0}\left[-\frac{1}{l}\right]_d^\infty$$

$$\boxed{W = \frac{q^2}{16\pi\epsilon_0 d}}$$

(e) The potential energy between the charge q and its image:

$$W = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{x}_1 - \mathbf{x}_2|}$$

$$W = -\frac{q^2}{8\pi\epsilon_0 d}$$

The potential energy is twice the work required to move the particle to infinity. The reason they don't match is because the image particle is not a real particle. We must remember that there are actually no fields within the conductor. The potential energy calculation above is counting the energy of the fields in the conductor, which don't actually exist.

(f) We now find the answer to part d in electron volts for an electron originally one angstrom from the surface.

$$W = \frac{q^2}{16\pi\epsilon_0 d}$$

$$W = \frac{(1\,\mathrm{e})^2}{16\pi(5.526\times10^7\,\mathrm{e/Vm})(10^{-10}\,\mathrm{m})}$$

$$\boxed{W = 3.6\,\mathrm{eV}}$$