C. S. BAIRD

Jackson 2.14 Homework Problem Solution

Dr. Christopher S. Baird, Fall 2012
University of Massachusetts Lowell

UMASS LOWELL

PROBLEM:

A variant of the preceding two-dimensional problem is a long hollow conducting cylinder of radius b that is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$.

(a) Solve by means of the series solution (2.71) and show that the potential inside the cylinder is

$$\Phi(\rho,\phi)=\frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{2n+1}$$

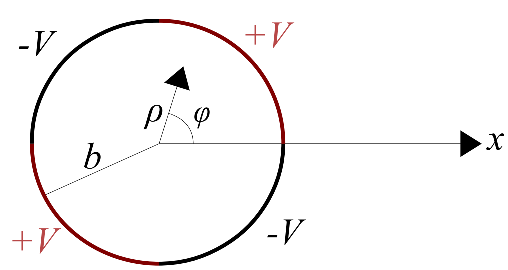
(b) Sum the series and show that

$$\Phi(\rho,\phi)=\frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2\sin 2\phi}{b^4-\rho^4}\right)$$

(c) Sketch the field lines and equipotentials

SOLUTION:

(a) Let us first sketch the problem and define our coordinate system:



The Laplace equation in polar coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

Attempting a solution of the form $\Phi(\rho,\phi)=R(\rho)\Psi(\phi)$ leads to:

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)=\nu^2 R(\rho) \quad \text{and} \quad \frac{\partial^2\Psi(\phi)}{\partial\phi^2}=-\nu^2\Psi(\phi)$$

which when solved leads to the general solution:

$$\Phi(\rho,\phi)=(a_0+b_0\ln\rho)(A_0+B_0\phi)+\sum_{\nu,\nu\neq 0}\left(a_\nu\rho^{\nu}+b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$

The Laplace equation contains two derivatives for each of the two coordinates, so that we will need four boundaries conditions to uniquely determine the four integration constants. The four boundary conditions are:

$$\Phi(\phi=0)=\Phi(\phi=2\pi)$$

$$\Phi(\rho=0)=\text{finite}$$

$$\Phi(\rho=b)=\begin{cases} V & \text{if } 0<\phi<\pi/2 \text{ or } \pi<\phi<3\pi/2 \\ -V & \text{if } \pi/2<\phi<\pi \text{ or } 3\pi/2<\phi<2\pi \end{cases}$$

Applying the first boundary conditions forces υ to be an integer which we relabel as m to designate this, and $B_0 = 0$.

Applying the second boundary condition leads to $b_\upsilon = 0$ and $b_0 = 0$. The terms in front can now be combined into a single constant and joined into the sum as the $m = 0$ case. Our solution thus reads:

$$\Phi(\rho,\phi)=\sum_{m=0}^{\infty}\rho^{m}\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)$$

We can combine the positive and negative m terms

$$\Phi(\rho,\phi)=\sum_{m=-\infty}^{\infty}A_m\rho^{|m|}e^{im\phi}$$

Apply the last boundary condition, which for now we state as $V(\phi)$

$$V(\phi)=\sum_{m=-\infty}^{\infty}A_m b^{|m|}e^{im\phi}$$

Multiple both sides by a complex exponential and integrate:

$$\int_0^{2\pi}V(\phi)e^{-im'\phi}d\phi=\sum_{m=-\infty}^{\infty}A_m b^{|m|}\int_0^{2\pi}e^{i(m-m')\phi}d\phi$$

Now recognize the integral on the right as the statement of orthogonality for complex exponentials so that:

$$\int_0^{2\pi}V(\phi)e^{-im'\phi}d\phi=\sum_{m=-\infty}^{\infty}A_m b^{|m|}2\pi\delta_{mm'}$$

Apply the delta and solve for A_m:

$$A_m = \frac{1}{2\pi b^{|m|}} \int_0^{2\pi} V(\phi) e^{-im\phi} d\phi$$

Now apply the actual boundary condition to this integral:

$$A_m = \frac{1}{2\pi b^{|m|}} \left[V \int_0^{\pi/2} e^{-im\phi} d\phi - V \int_{\pi/2}^{\pi} e^{-im\phi} d\phi + V \int_{\pi}^{3\pi/2} e^{-im\phi} d\phi - V \int_{3\pi/2}^{2\pi} e^{-im\phi} d\phi \right]$$

$$A_m = \frac{V}{(-im) 2\pi b^{|m|}} \left[(e^{-im(\pi/2)} - e^{-im(0)}) - (e^{-im(\pi)} - e^{-im(\pi/2)}) + (e^{-im(3\pi/2)} - e^{-im(\pi)}) - (e^{-im(2\pi)} - e^{-im(3\pi/2)}) \right]$$

$$A_m = \frac{V}{(-im)\pi b^{|m|}} \left[e^{-im\pi/2} - 1 - e^{-im\pi} + e^{-im3\pi/2} \right]$$

A quick calculation reveals that if m is odd, A_m is always zero. With m even we have:

$$A_m = \frac{2V}{(-im)\pi b^{|m|}} \left[(-1)^{m/2} - 1 \right]$$

So that the final solution is:

$$\Phi(\rho, \phi) = \sum_{m=-\infty,\ \text{even}}^{\infty} \frac{2V}{im\pi} \left[1 - (-1)^{m/2} \right] \left(\frac{\rho}{b} \right)^{|m|} e^{im\phi}$$

Let us break up the positive and negative m cases.

$$\Phi(\rho, \phi) = \sum_{m=0,\text{even}}^{\infty} \frac{2V}{im\pi} \left(\frac{\rho}{b} \right)^m \left[1 - (-1)^{m/2} \right] \left[e^{im\phi} - e^{-im\phi} \right]$$

$$\Phi(\rho, \phi) = \sum_{m=0,\text{even}}^{\infty} \frac{4V}{m\pi} \left(\frac{\rho}{b} \right)^m \left[1 - (-1)^{m/2} \right] \sin(m\phi)$$

For m = 0, 4, 8, … this also disappears, so that only the terms remain for m = 2, 6, 10, 14... for which the solution reduces to:

$$\Phi(\rho, \phi) = \sum_{m=2,6,10\ldots}^{\infty} \frac{8V}{m\pi} \left(\frac{\rho}{b} \right)^m \sin(m\phi)$$

Now make a change for variables $m = 4n + 2$ where n = 0, 1, 2...

$$\boxed{\Phi(\rho, \phi) = \frac{4V}{\pi} \sum_{n=0}^{\infty} \left(\frac{\rho}{b} \right)^{4n+2} \frac{\sin[(4n+2)\phi]}{2n+1}}$$

(b) The sine function is the imaginary part of a complex exponential

$$\Phi(\rho,\phi)=\Im\left[\frac{8V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}e^{i\phi}\right)^{4n+2}\frac{1}{4n+2}\right]$$

Define $Z=\frac{\rho}{b}e^{i\phi}$ so that we end up with the form:

$$\Phi(\rho,\phi)=\Im\left[\frac{8V}{\pi}\sum_{m=2,6,10...}^{\infty}\frac{Z^m}{m}\right]$$

This looks similar to the expansion of the logarithm:

$$\ln(1+Z)=\sum_{n=1}^{\infty}\frac{Z^n}{n}(-1)^{n-1}$$

Let us try to get them to match. Flip the sign of Z to find:

$$\ln(1-Z)=-\sum_{n=1}^{\infty}\frac{Z^n}{n}$$

$$\ln(1+Z)+\ln(1-Z)=-2\sum_{n=2,4,6...}^{\infty}\frac{Z^n}{n}$$

$$\ln[(1+Z)(1-Z)]=-2\sum_{n=2,4,6...}^{\infty}\frac{Z^n}{n}$$

Replace Z in this expression by Z^2 to find:

$$-\frac{1}{2}\ln[(1+Z^2)(1-Z^2)]=+2\sum_{n=4,8,12...}^{\infty}\frac{Z^n}{n}$$

$$\ln[(1+Z)(1-Z)]-\frac{1}{2}\ln[(1+Z^2)(1-Z^2)]=-2\sum_{n=2,6,10...}^{\infty}\frac{Z^n}{n}$$

Applying this relation to our problem and simplifying, we find:

$$\Phi(\rho,\phi)=\frac{2V}{\pi}\Im\left[\ln\left[\frac{1+Z^2}{1-Z^2}\right]\right]$$

Using $\Im[\ln z]=\operatorname{Arg} z$

$$\Phi(\rho,\phi)=\frac{2V}{\pi}\operatorname{Arg}\left[\frac{1+Z^2}{1-Z^2}\right]$$

Using: $\operatorname{Arg} z = \tan^{-1}\left((-i)\left(\frac{z-z^*}{z+z^*}\right)\right)$

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left((-i)\left(\frac{(\frac{1+Z^2}{1-Z^2}) - (\frac{1+Z^2}{1-Z^2})^*}{(\frac{1+Z^2}{1-Z^2}) + (\frac{1+Z^2}{1-Z^2})^*}\right)\right)$$

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left((-i)\left(\frac{Z^2 - Z^{*2}}{1 - Z^2 Z^{*2}}\right)\right)$$

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left((-i)\left(\frac{Z^2 - Z^{*2}}{1 - Z^2 Z^{*2}}\right)\right)$$

Substitute back in $Z = \frac{\rho}{b} e^{i\phi}$

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left((-i)\left(\frac{\frac{\rho^2}{b^2}e^{i2\phi} - \frac{\rho^2}{b^2}e^{-i2\phi}}{1 - \frac{\rho^2}{b^2}e^{i2\phi}\frac{\rho^2}{b^2}e^{-i2\phi}}\right)\right)$$

$$\boxed{\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2 \sin(2\phi)}{b^4 - \rho^4}\right)}$$

(c) A numerical plot of this solution can be made. We can also sketch the equipotentials and field lines: