Jackson 2.13 Homework Problem Solution

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PROBLEM:

(a) Two halves of a long hollow conducting cylinder of inner radius *b* are separated by small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(\rho,\phi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

where ϕ is measured from a plane perpendicular to the plane through the gap.

(b) Calculate the surface-charge density on each half of the cylinder.

SOLUTION:

Due to the symmetry of the problem, it is apparent that the solution will be best expressed in cylindrical coordinates. Additionally, because the solution will be independent of the *z* coordinate, the problem reduces to the two dimensions of polar coordinates (ρ,ϕ). Because the problem contains no charge, the problem simplifies down to solving the Laplace equation $\nabla^2\Phi=0$ in polar coordinates and applying the boundary condition $\Phi(\rho=b,\phi)=V(\phi)$ where:

$$V(\phi)=\begin{Bmatrix} V_1 & \text{if } \pi/2>\phi>3\pi/2 \\ V_2 & \text{if } \pi/2<\phi<3\pi/2 \end{Bmatrix}$$

The Laplace equation in polar coordinates is:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

Separation of variables leads to the general solution:

$$\Phi(\rho,\phi)=(a_0+b_0\ln\rho)(A_0+B_0\phi)+\sum_{\nu,\nu\neq 0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$

We desire a valid solution at the origin, which is only possible if $b_0 = 0$ and $b_\nu = 0$ so that the solution becomes:

$$\Phi(\rho,\phi)=A_0+B_0\phi+\sum_{\nu,\nu\neq 0}\rho^\nu\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$

We desire a single, valid solution over the full angular range, so the single-value requirement means $\Phi(\rho,\phi)=\Phi(\rho,\phi+2\pi)$. When we apply this, we get:

$$A_0+B_0\phi+\sum_{\nu,\nu\neq 0}\rho^{\nu}\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)=A_0+B_0(\phi+2\pi)+\sum_{\nu,\nu\neq 0}\rho^{\nu}\left(A_\nu e^{i\nu(\phi+2\pi)}+B_\nu e^{-i\nu(\phi+2\pi)}\right)$$

Which leads to $B_0 = 0$ and $\upsilon = n$ where $n = 1, 2, \ldots$ We now have:

$$\Phi(\rho,\phi)=A_0+\sum_{n=1}^{\infty}\rho^n\left(A_n e^{in\phi}+B_n e^{-in\phi}\right)$$

Now apply the last boundary condition $\Phi(\rho=b,\phi)=V(\phi)$

$$V(\phi)=A_0+\sum_{n=1}^{\infty}b^n\left(A_n e^{in\phi}+B_n e^{-in\phi}\right)\quad (Eq.\ 1)$$

Let us first find the A_0 term. Integrate both sides over the full angular sweep.

$$\int_0^{2\pi}V(\phi)\,d\phi=\int_0^{2\pi}A_0\,d\phi+\sum_{n=1}^{\infty}b^n\left(A_n\int_0^{2\pi}e^{in\phi}\,d\phi+B_n\int_0^{2\pi}e^{-in\phi}\,d\phi\right)$$

$$\int_{-\pi/2}^{\pi/2}V_1\,d\phi+\int_{\pi/2}^{3\pi/2}V_2\,d\phi=A_0 2\pi$$

$$V_1\pi+V_2\pi=A_0 2\pi$$

$$\boxed{A_0=\frac{V_1+V_2}{2}}$$

Let us now find the A_n coefficients. Multiply ($Eq.$ 1) on both sides by $e^{-in'\phi}$ and integrate over all angles ϕ :

$$\int_0^{2\pi}V(\phi)e^{-in'\phi}\,d\phi=A_0\int_0^{2\pi}e^{-in'\phi}\,d\phi+\sum_{n=1}^{\infty}b^n\left(A_n\int_0^{2\pi}e^{i(n-n')\phi}\,d\phi+B_n\int_0^{2\pi}e^{-i(n+n')\phi}\,d\phi\right)$$

Use the orthonormality condition $\int_0^{2\pi}e^{i(k-k')x}\,dx=2\pi\,\delta_{k,k'}$

$$\int_0^{2\pi}d\phi\,V(\phi)e^{-in\phi}=2\pi b^n A_n$$

$$A_n=\frac{1}{2\pi b^n}\int_0^{2\pi}d\phi\,V(\phi)e^{-in\phi}$$

Plug in the explicit form of the potential on the boundary which breaks the integral into two parts:

$$A_n = \frac{1}{2\pi b^n}\left[V_1 \int_{-\pi/2}^{\pi/2} d\phi\, e^{-in\phi} + V_2 \int_{\pi/2}^{3\pi/2} d\phi\, e^{-in\phi}\right]$$

$$A_n = \frac{1}{2\pi b^n}\left[V_1 \left[\frac{e^{-in\phi}}{-in}\right]_{-\pi/2}^{\pi/2} + V_2 \left[\frac{e^{-in\phi}}{-in}\right]_{\pi/2}^{3\pi/2}\right]$$

$$\boxed{A_n = \frac{(-1)^{(n+1)/2}}{-n\pi b^n}\left[V_1 - V_2\right]} \quad \text{and} \quad A_n = 0 \quad \text{if} \quad n = \text{even}$$

Let us now solve for the B_n coefficients. Take (*Eq.* 1) again and this time multiply by $e^{in'\phi}$ and integrate over all ϕ :

$$\int_0^{2\pi} V(\phi) e^{in'\phi} d\phi = A_0 \int_0^{2\pi} e^{in'\phi} d\phi + \sum_{n=1}^{\infty} b^n \left(A_n \int_0^{2\pi} e^{i(n+n')\phi} d\phi + B_n \int_0^{2\pi} e^{i(n'-n)\phi} d\phi\right)$$

Use the orthonormality condition $\int_0^{2\pi} e^{i(k-k')x} dx = 2\pi\,\delta_{k,k'}$

$$\int_0^{2\pi} V(\phi) e^{in\phi} d\phi = b^n B_n 2\pi$$

$$B_n = \frac{1}{2\pi b^n} \int_0^{2\pi} V(\phi) e^{in\phi} d\phi$$

Plug in the explicit form of the potential on the boundary which breaks the integral into two parts:

$$B_n = \frac{1}{2\pi b^n}\left[V_1 \int_{-\pi/2}^{\pi/2} d\phi\, e^{in\phi} + V_2 \int_{\pi/2}^{3\pi/2} d\phi\, e^{in\phi}\right]$$

$$B_n = \frac{1}{2\pi b^n}\left[V_1 \left[\frac{e^{in\phi}}{in}\right]_{-\pi/2}^{\pi/2} + V_2 \left[\frac{e^{in\phi}}{in}\right]_{\pi/2}^{3\pi/2}\right]$$

$$\boxed{B_n = \frac{(-1)^{(n+1)/2}}{-n\pi b^n}\left[V_1 - V_2\right]} \quad \text{and} \quad B_n = 0 \quad \text{if} \quad n = \text{even}$$

Now that we have found all of the coefficients, the solution is determined:

$$\Phi(\rho,\phi) = \frac{V_1 + V_2}{2} + \sum_{n=1,\text{odd}}^{\infty} \rho^n \frac{(-1)^{(n+1)/2}}{-n\pi b^n}\left[V_1 - V_2\right]\left(e^{in\phi} + e^{-in\phi}\right)$$

$$\Phi(\rho,\phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{2}\frac{4}{\pi} \sum_{n=1,\text{odd}}^{\infty} \frac{(-1)^{(n+1)/2}\rho^n}{-n b^n} \cos(n\phi)$$

$$\Phi(\rho,\phi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{2}\frac{4}{\pi}\Re\sum_{n=1,\text{odd}}^{\infty}\frac{(-1)(-1)^{(n+1)/2}}{n}\left(\frac{\rho e^{i\phi}}{b}\right)^n$$

Now we recognize the Taylor expansion of the arctan: $\tan^{-1}(x)=\sum_{n=1,\text{odd}}^{\infty}\frac{(-1)(-1)^{(n+1)/2}}{n}x^n$

$$\Phi(\rho,\phi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{2}\frac{4}{\pi}\Re\left[\tan^{-1}\left(\frac{\rho e^{i\phi}}{b}\right)\right]$$

Using the identity $\tan^{-1}(z)=\frac{i}{2}[\ln(1-iz)-\ln(1+iz)]$ and expanding the complex number z in its components, we can prove the identity:

$$\Re\left[\tan^{-1}(z)\right]=\frac{1}{2}\tan^{-1}\left(\frac{2\Re(z)}{1-|z|^2}\right)$$

We now use this identity:

$$\boxed{\Phi(\rho,\phi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)}$$

(b) Calculate the surface-charge density on each half of the cylinder.

As derived earlier using a Gaussian pillbox surface, the surface-charge density on a conductor is related to the potential according to:

$$\sigma=\left[-\epsilon_0\frac{d\Phi}{dn}\right]_{n=a}$$

We assume that the problem is seeking the charge density on the inside surface of the conductor because that is where we know the potential. In that case, the normal to the inside surface of the conductor points in the opposite direction as the radial dimension, so that $n = -\rho$.

$$\sigma=\left[\epsilon_0\frac{d\Phi}{d\rho}\right]_{\rho=b}$$

$$\sigma=\left[\epsilon_0\frac{d}{d\rho}\left(\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)\right)\right]_{\rho=b}$$

$$\sigma=\left[\epsilon_0\frac{V_1-V_2}{\pi}\left(\frac{1}{1+\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)^2}\right)\frac{d}{d\rho}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)\right]_{\rho=b}$$

$$\sigma = \left[\epsilon_0 \frac{V_1 - V_2}{\pi} \left(\frac{2\,b\cos\phi\left(b^2 + \rho^2\right)}{(b^2 - \rho^2)^2 + \left(2\,b\,\rho\cos\phi\right)^2} \right) \right]_{\rho = b}$$

$$\boxed{\sigma = \epsilon_0 \frac{V_1 - V_2}{\pi\, b\cos\phi}}$$