Jackson 2.12 Homework Problem Solution

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PROBLEM:

Starting with the series solution (2.71) for the two-dimensional potential problem with the potential specified on the surface of a cylinder of radius *b*, evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential inside the cylinder in the form of Poisson's integral:

$$\Phi(\rho,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity?

SOLUTION:

When the potentials and charges are uniform in the *z*-direction, as is the case in this problem with the cylinder, the system reduces down to a two-dimensional problem. The circular cross-section of the cylinder dictates that polar coordinates are the most fitting two-dimensional coordinate system to use. There are not charges involved, so we need to solve the Laplace equation. The series solution to the Laplace equation in polar coordinates was found in Jackson to be (Eq. 2.71):

$$\Phi(\rho,\phi)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}a_n\rho^n\sin(n\phi-\alpha_n)+\sum_{n=1}^{\infty}b_n\rho^{-n}\sin(n\phi+\beta_n)$$

Because this is a solution to a two-dimensional second order differential equation (the Laplace equation), there must be four undetermined coefficients (or sets of coefficients for series solutions). We therefore need four boundary conditions; or a boundary condition at the maximum and minimum of each dimension. For this problem the boundary conditions are:

(1) angular minimum: $\Phi(\phi=0)=\Phi(\phi=2\pi)$
(2) angular maximum: $\Phi(\phi=2\pi)=\Phi(\phi=0)$
(3) radial minimum: $\Phi(\rho=0)=\text{finite}$
(4) radial maximum: $\Phi(\rho=b)=V(\phi)$

Boundary conditions (1) and (2) are the same boundary conditions ensuring periodicity because the full angular sweep is included in the region of interest. Applying boundary conditions (1) and (2) leads to the coefficient multiplied against phi becoming an integer. These boundary conditions have already been applied in the solution given in Jackson Eq. 2.71. With two boundary conditions left, we should only have two undetermined sets of coefficients. But Eq. 2.71 seems to contain four independent undetermined coefficients. In reality, they are not all independent and will be taken care of as we proceed.

When we apply boundary condition (3) we see right away that:

$b_n = 0$ and $b_0 = 0$

so that the solution becomes:

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi - \alpha_n)$$

Because a_n and α_n are arbitrary at this point, we can redefine them as we want to get this general solution into a more useful form. Take a b^{-n} out of the a_n constants.

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{\rho}{b}\right)^n \sin(n\phi - \alpha_n)$$

Use $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2i} \left(\frac{\rho}{b}\right)^n [e^{-i\alpha_n} e^{in\phi} - e^{i\alpha_n} e^{-in\phi}]$$

Redefine constants:

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} \left(\frac{\rho}{b}\right)^n [c_n e^{in\phi} + d_n e^{-in\phi}]$$

Both terms (as well as a_0) can be combined by letting the summation index extend to negative numbers:

$$\Phi(\rho,\phi) = \sum_{n=-\infty}^{\infty} c_n \left(\frac{\rho}{b}\right)^{|n|} e^{in\phi}$$

Apply boundary condition (4) to find:

$$V(\phi) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi}$$

Multiple both sides by a complex exponential and integrate:

$$\int_0^{2\pi} V(\phi) e^{-in'\phi} d\phi = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-n')\phi} d\phi$$

Now recognize the integral on the right as the statement of orthogonality for complex exponentials so that:

$$\int_0^{2\pi} V(\phi) e^{-in'\phi} d\phi = \sum_{n=-\infty}^{\infty} c_n 2\pi \delta_{nn'}$$

Apply the delta and solve for c_n:

$$\boxed{c_n = \frac{1}{2\pi}\int_0^{2\pi} V(\phi) e^{-in\phi}\, d\phi}$$

Our final solution becomes:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' V(\phi') \sum_{n=-\infty}^{\infty} \left(\frac{\rho}{b}\right)^{|n|} e^{in(\phi-\phi')}$$

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' V(\phi') \left[-1 + \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{n} e^{in(\phi-\phi')} + \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{n} e^{-in(\phi-\phi')}\right]$$

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' V(\phi') \left[-1 + \sum_{n=0}^{\infty} \left[\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right]^{n} + \sum_{n=0}^{\infty} \left[\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}\right]^{n}\right]$$

Use $\sum_0^{\infty} r^n = \frac{1}{1-r}$

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' V(\phi') \left[-1 + \frac{1}{1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}} + \frac{1}{1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}}\right]$$

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' V(\phi') \left[\frac{\left(\left(1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}\right) + \left(1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right) - \left(1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right)\left(1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}\right)\right)}{\left(1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right)\left(1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}\right)}\right]$$

$$\boxed{\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi') \frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\, d\phi'}$$

If instead, we want to find the potential in the region external to the cylinder, we swap b and ρ to find:

$$\boxed{\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi') \frac{\rho^2-b^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\, d\phi'}$$