# Jackson 2.11 Homework Problem Solution

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**PROBLEM:**

A line charge with linear charge density $\tau$ is placed parallel to, and a distance $R$ away from, the axis of a conducting cylinder of radius $b$ held at fixed voltage such that the potential vanishes at infinity. Find

(a) the magnitude and position of the image charge(s);

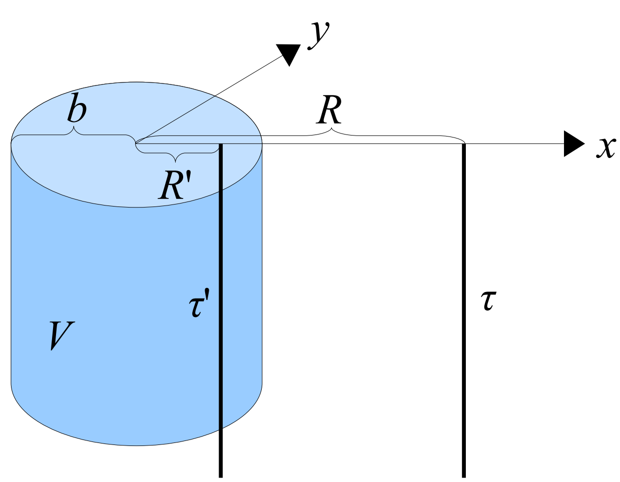
(b) the potential at any point (expressed in polar coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the $x$ axis), including the asymptotic form far from the cylinder;

(c) the induced surface-charge density, and plot it as a function of angle for $R/b = 2, 4$ in units of $\tau/2\pi b$

(d) the force per unit length on the line charge

**SOLUTION:**

This problem is similar to a point charge next to sphere. Let us place an image line charge $\tau'$ inside the cylinder at $R'$ as shown in the diagram below.




The electric field surrounding one wire without anything else present is found by drawing a cylindrical Gaussian surface around the line charge and using Gauss's Law. Due to the symmetry, the electric field is parallel to the surface normal and constant over the Gaussian surface. It can thus be taken out of the integral.

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da=\frac{1}{\epsilon_0}q_{enc}$$

$$E\oint_S da=\frac{1}{\epsilon_0}\tau L$$

$$E\,2\pi r\,L=\frac{1}{\epsilon_0}\tau L$$

$$\mathbf{E}=\frac{\tau}{2\pi\epsilon_0 r}\hat{\mathbf{r}}$$

Use the definition of the potential and solve for the potential

$$\mathbf{E}=-\nabla\Phi$$

$$\frac{d\Phi}{dr}=-\frac{\tau}{2\pi\epsilon_0 r}$$

$$\Phi=-\frac{\tau}{2\pi\epsilon_0}\ln(r)+A$$

Set the arbitrary integration constant to $A=\frac{\tau}{2\pi\epsilon_0}\ln(B)$ and use the laws of logarithms:

$$\Phi=\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{B^2}{r^2}\right)$$ (The potential due to one line charge)

The total potential is now found by including both the line charge and the image line charge:

$$\Phi=\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{B^2}{r_1^2}\right)+\frac{\tau'}{4\pi\epsilon_0}\ln\left(\frac{B^2}{r_2^2}\right)$$

The variables $r_1$ and $r_2$ are the distance from the respective wires to the observation point. We must now express them in terms of the cylindrical coordinates $(\rho,\phi,z)$:

$$\Phi=\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{B^2}{\rho^2+R^2-2\rho R\cos\phi}\right)+\frac{\tau'}{4\pi\epsilon_0}\ln\left(\frac{B^2}{\rho^2+R'^2-2\rho R'\cos\phi}\right)$$

Apply the boundary condition $\Phi(\rho\to\infty)=0$

$$0=\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{B^2}{\rho^2+R^2-2\rho R\cos\phi}\right)+\frac{\tau'}{4\pi\epsilon_0}\ln\left(\frac{B^2}{\rho^2+R'^2-2\rho R'\cos\phi}\right)$$

$$(\rho^2+R^2-2\rho R\cos\phi)^{\tau}(\rho^2+R'^2-2\rho R'\cos\phi)^{\tau'}=(B^2)^{\tau+\tau'}$$

As $\rho$ approaches infinity, only the highest power of $\rho$ will survive and all other terms will approach zero by comparison:

$$(\rho^2+0-0)^{\tau}(\rho^2+0-0)^{\tau'}=(B^2)^{\tau+\tau'}$$

$$(\rho^2)^{\tau+\tau'}=(B^2)^{\tau+\tau'}$$

$$(\tau+\tau')(\ln(\rho^2)-\ln(B^2))=0$$

$$(\tau+\tau')=0$$

$$\boxed{\tau'=-\tau}$$

This makes sense because for the potential to be zero at infinity, the total charge should be zero. The image charge cancels out the line charge at large distances. The solution now becomes:

$$\Phi=\frac{\tau}{4\pi\epsilon_0}\left[\ln\left(\frac{B^2}{\rho^2+R^2-2\rho R\cos\phi}\right)-\ln\left(\frac{B^2}{\rho^2+R'^2-2\rho R'\cos\phi}\right)\right]$$

$$\Phi=\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{\rho^2+R'^2-2\rho R'\cos\phi}{\rho^2+R^2-2\rho R\cos\phi}\right)$$

Apply the boundary condition $\Phi(\rho=b)=V$

$$V=\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{b^2+R'^2-2bR'\cos\phi}{b^2+R^2-2bR\cos\phi}\right)$$

$$e^{\left(\frac{4\pi\epsilon_0 V}{\tau}\right)}=\frac{b^2+R'^2-2bR'\cos\phi}{b^2+R^2-2bR\cos\phi}$$

$$(b^2+R^2-2bR\cos\phi)e^{\left(\frac{4\pi\epsilon_0 V}{\tau}\right)}=b^2+R'^2-2bR'\cos\phi$$

$$(b^2+R^2)e^{\left(\frac{4\pi\epsilon_0 V}{\tau}\right)}-b^2-R'^2=\left[-2bR'+2bRe^{\left(\frac{4\pi\epsilon_0 V}{\tau}\right)}\right]\cos\phi$$

This must be true for all $\phi$ so that the both sides of the equation are independent and thus equal to a constant. The constant must be zero to accommodate the case of $\phi=\pi/2$ .

$$(b^2+R^2)e^{\left(\frac{4\pi\epsilon_0 V}{\tau}\right)}-b^2-R'^2=0 \quad \text{and} \quad 0=-2bR'+2bRe^{\left(\frac{4\pi\epsilon_0 V}{\tau}\right)}$$

$$(b^2+R^2)e^{\left(\frac{4\pi\epsilon_0 V}{\tau}\right)}-b^2-R'^2=0 \quad \text{and} \quad \frac{R'}{R}=e^{\left(\frac{4\pi\epsilon_0 V}{\tau}\right)}$$

We can use these two equations to eliminate the dependence on $V$ and make the solution more general.

$$(b^2+R^2)\frac{R'}{R}-b^2-R'^2=0$$

$$R'^2-(b^2+R^2)\frac{R'}{R}+b^2=0$$

$$\boxed{R'=\frac{b^2}{R}}$$

(b) the potential at any point (expressed in polar coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the $x$ axis), including the asymptotic form far from the cylinder;

Plugging in the image charge magnitude and location as found above, the solution to the potential now becomes:

$$\boxed{\Phi=\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{\rho^2+b^4/R^2-2\rho(b^2/R)\cos\phi}{\rho^2+R^2-2\rho R\cos\phi}\right)}$$

To get the asymptotic form, we put the term in parentheses in a form that is easy to expand:

$$\Phi=\frac{\tau}{4\pi\epsilon_0}\ln\left(1-1+\frac{\rho^2+b^4/R^2-2\rho(b^2/R)\cos\phi}{\rho^2+R^2-2\rho R\cos\phi}\right)$$

$$\Phi=\frac{\tau}{4\pi\epsilon_0}\ln\left(1+\frac{-(R^4-b^4)(1/R^2)+(R^2-b^2)2(\rho/R)\cos\phi}{(\rho^2+R^2-2\rho R\cos\phi)}\right)$$

Use the expansion $\ln(1+x)=x-x^2/2+x^3/3+...$

$$\Phi=\frac{\tau}{4\pi\epsilon_0}[\left(\frac{-(R^4-b^4)(1/R^2)+(R^2-b^2)2(\rho/R)\cos\phi}{(\rho^2+R^2-2\rho R\cos\phi)}\right) -1/2\left(\frac{-(R^4-b^4)(1/R^2)+(R^2-b^2)2(\rho/R)\cos\phi}{(\rho^2+R^2-2\rho R\cos\phi)}\right)^2+...]$$

"Far away" from the cylinder is defined as $\rho >> b$ and "far away" from the line charge is defined as $\rho >> R$ so that we can drop all the higher order terms in the expansion

$$\Phi=\frac{\tau}{4\pi\epsilon_0}\frac{-(R^4-b^4)(1/R^2)+(R^2-b^2)2(\rho/R)\cos\phi}{(\rho^2+R^2-2\rho R\cos\phi)}$$

Similarly, we can drop all but the highest term in the numerator and denominator.

$$\boxed{\Phi=\frac{\tau}{2\pi\epsilon_0}\frac{(R^2-b^2)}{\rho R}\cos\phi}$$

(c) the induced surface-charge density, and plot it as a function of angle for $R/b$ = 2, 4 in units of $\tau/2\pi b$

As shown previously, the surface-charge density on a conductor is found using Gauss's Law to be:

$$\left[E_n=\frac{1}{\epsilon_0}\sigma\right]_{n=n_0}$$

$$\sigma=\left[-\epsilon_0\frac{\partial\Phi}{\partial n}\right]_{n=n_0}$$

The normal to the conductor's surface is just in the cylindrical radial direction:

$$\sigma=\left[-\epsilon_0\frac{\partial\Phi}{\partial\rho}\right]_{\rho=b}$$

$$\sigma=\left[-\epsilon_0\frac{\partial}{\partial\rho}\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{\rho^2+b^4/R^2-2\rho(b^2/R)\cos\phi}{\rho^2+R^2-2\rho R\cos\phi}\right)\right]_{\rho=b}$$

$$\sigma=\left[-\epsilon_0\frac{\tau}{4\pi\epsilon_0}\frac{\partial}{\partial\rho}\left[\ln\left(\rho^2+b^4/R^2-2\rho(b^2/R)\cos\phi\right)-\ln\left(\rho^2+R^2-2\rho R\cos\phi\right)\right]\right]_{\rho=b}$$

$$\sigma=\left[-\epsilon_0\frac{\tau}{4\pi\epsilon_0}\left[\frac{2\rho-2(b^2/R)\cos\phi}{\rho^2+b^4/R^2-2\rho(b^2/R)\cos\phi}-\frac{2\rho-2R\cos\phi}{\rho^2+R^2-2\rho R\cos\phi}\right]\right]_{\rho=b}$$

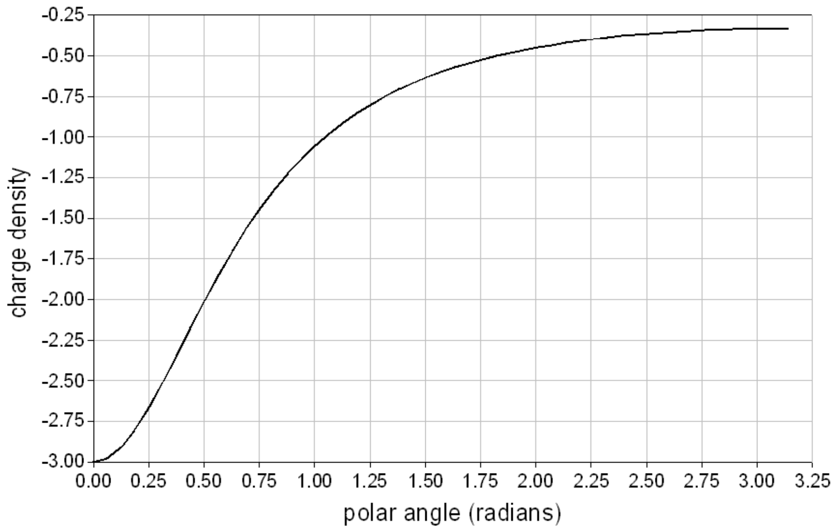
$$\boxed{\sigma=\frac{\tau}{2\pi b}\left[\frac{1-(R/b)^2}{1+(R/b)^2-2(R/b)\cos\phi}\right]}$$

For $R/b$ = 2

$$\sigma=\frac{\tau}{2\pi b}\left[\frac{-3}{5-4\cos\phi}\right]$$

In units of ($\tau/2\pi b$) this becomes:

$$\sigma=\frac{-3}{5-4\cos\phi}$$

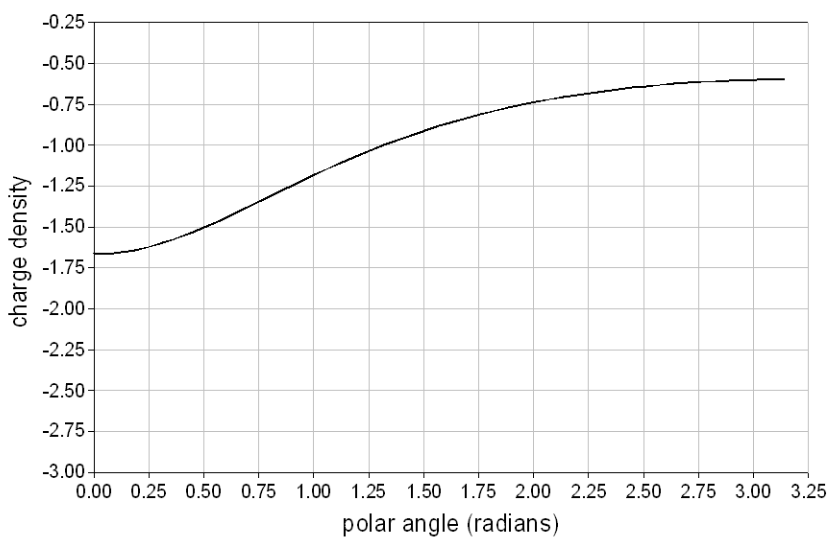



For $R/b = 4$

$$\sigma = \frac{\tau}{2\pi\, b}\left[\frac{-15}{17 - 8\cos\phi}\right]$$

In units of $(\tau/2\pi b)$ this becomes:

$$\sigma = \frac{-15}{17 - 8\cos\phi}$$



(d) the force per unit length on the line charge

The electric field felt at some point at a distance $d$ from the image line charge due to the image line charge is:

$$\mathbf{E}=\frac{\tau'}{2\pi\epsilon_0 d}\hat{\boldsymbol{\rho}}$$

The image charge is known to be $\tau'=-\tau$ and the distance $d$ is just the distance between the image charge and the line charge, $d=R-R'=R-\frac{b^2}{R}$, so that

$$\mathbf{E}=-\frac{\tau}{2\pi\epsilon_0(R-b^2/R)}\hat{\boldsymbol{\rho}}$$

The force is the charge being acted upon times the electric field:

$$\mathbf{F}=q\,\mathbf{E}$$

$\mathbf{F}=(\tau L)\mathbf{E}$ where $L$ is some length along the line charge

$$\mathbf{F}=(\tau L)\left(-\frac{\tau}{2\pi\epsilon_0(R-b^2/R)}\hat{\boldsymbol{\rho}}\right)$$

Force per unit length: $\boxed{\frac{\mathbf{F}}{L}=-\frac{\tau^2 R}{2\pi\epsilon_0(R^2-b^2)}\hat{\boldsymbol{\rho}}}$ (The force is attractive)