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Jackson 1.9 Homework Problem Solution

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PROBLEM:

Calculate the attractive force between conductors in the parallel plate capacitor (Problem 1.6a: area A , separated by distance d) and the parallel cylinder capacitor (Problem 1.7: radii a_1 and a_2 , separated by distance d) for

- (a) fixed charges on each conductor
- (b) fixed potential difference between conductors.

SOLUTION:

(a) Parallel Plates

Because the plates are large, flat, and close, we can neglect fringe effects. If one plate has total charge $+Q$ and the other plate has total charge $-Q$, then they each have a charge density

$$\rho = \frac{Q}{A} \delta(z-d) \quad \text{and} \quad \rho = -\frac{Q}{A} \delta(z) \quad \text{for } 0 < x < A^{1/2} \text{ and } 0 < y < A^{1/2}$$

The total force on the positive plate due to the other plate is the sum of the force on all of its parts, which mathematically takes the form of an integral:

$$\mathbf{F} = \int \rho(\mathbf{x}) \mathbf{E}(\mathbf{x}) d\mathbf{x}$$

Here \mathbf{E} is not the total electric field, but the electric field due to the negative plate

Expand in Cartesian coordinates:

$$\mathbf{F} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, y, z) \mathbf{E}(x, y, z) dx dy dz$$

Let us find the force on the positively charged plate placed at $z = d$ by substituting in its charge density:

$$\mathbf{F} = \frac{Q}{A} \int_0^{\sqrt{A}} \int_0^{\sqrt{A}} \mathbf{E}(x, y, d) dx dy$$

Because we are neglecting fringe effects (in effect we are assuming that the plates are large enough that they appear infinite when compared to the rest of the system), the symmetry of the system requires that the electric field can only be a function of z . The remaining integrals thus reduce down to the area of the plate, which cancels the area in the denominator.

$$\mathbf{F} = Q \mathbf{E}(z=d)$$

The electric field can be found by placing a Gaussian pillbox around a section of the each plate, as is demonstrated in detail elsewhere. Because the plate is thin, both top and bottom of the Gaussian surface have electric field lines crossing and contribute:

$$2 E_n = \frac{\sigma}{\epsilon_0}$$

$$E_n = \frac{\pm Q}{2 A \epsilon_0}$$

We must be careful to realize that the normal of the inside of the positive plate points towards $-z$ and the normal of the inside of the negative plate points towards $+z$.

$$\mathbf{E}_{\text{pos plate}} = \frac{Q}{2 A \epsilon_0} (-\hat{\mathbf{z}})$$

$$\mathbf{E}_{\text{neg plate}} = \frac{(-Q)}{2 A \epsilon_0} \hat{\mathbf{z}}$$

Because of the symmetry, the field is constant and uniform across that gap so that

$$\mathbf{E} = \mathbf{E}_{\text{pos plate}} + \mathbf{E}_{\text{neg plate}}$$

$$\mathbf{E} = \frac{-Q}{A \epsilon_0} \hat{\mathbf{z}}$$

But for the purposes of calculating the force, we do not use the total field, rather we use the field due to the negative plate because it is what acts on the positive plate.

$$\mathbf{F} = Q \mathbf{E}_{\text{neg plate}}$$

$$\boxed{\mathbf{F} = \frac{-Q^2}{2 A \epsilon_0} \hat{\mathbf{z}}}$$

The right plate experiences a force in the $-z$ direction, towards the other plate and is therefore attracted to it, as we should expect because opposites attract.

Parallel Cylinders:

Because the distance between the cylinders is large compared to their radii, we can approximate them as infinitely thin wires for the purpose of calculating the fields. Place cylinder 1 at the origin with positive charge per unit length $+\lambda$ and cylinder 2 at $x = d$ with negative charge per unit length $-\lambda$. If we want to find the force on cylinder 2, we only need to find the field due to cylinder 1 because cylinder 2 will not exert a net force on itself.

The field due to cylinder 1 can be found by placing a cylindrical Gaussian surface around it and taking

advantage of symmetry:

$$\oint \mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{q_{enc}}{\epsilon_0}$$

$$l \int \mathbf{E} \cdot \hat{\mathbf{r}} r d\phi = \frac{q_{enc}}{\epsilon_0}$$

$$E_r 2\pi r = \left(\frac{q_{enc}}{l} \right) \frac{1}{\epsilon_0}$$

Recognize q_{enc}/l as the charge per unit length λ :

$$\mathbf{E} = \frac{\lambda}{2\pi r \epsilon_0} \hat{\mathbf{r}}$$

$$\mathbf{F} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, y, z) \mathbf{E}(x, y, z) dx dy dz$$

$$\mathbf{F} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(y) \delta(x-d) (-\lambda) \mathbf{E}(x, y, z) dx dy dz$$

$$\frac{\mathbf{F}}{l} = -\lambda \mathbf{E}(d, 0)$$

$$\boxed{\frac{\mathbf{F}}{l} = \frac{-\lambda^2}{2\pi d \epsilon_0} \hat{\mathbf{r}}}$$

Cylinder 2 experiences a force in the $-r$ direction, towards the origin, where cylinder 1 resides.

(b) Parallel Plates:

A fixed potential difference V is maintained across the capacitor as opposed to fixed charges. Potential is related to electric field according to:

$$\mathbf{E} = -\nabla \Phi$$

For a one dimensional field, this becomes:

$$E = \frac{d\Phi}{dz}$$

For a uniform field this becomes:

$$E = \frac{V}{d} \quad \text{where } d \text{ is the distance between the two points where the potential is measured.}$$

The contribution from just one plate is $E_{\text{neg plate}} = \frac{V}{2d}$

We found previously the relationship between the total electric field and the charge on one plate to be:

$$E = \frac{Q}{A\epsilon_0}$$

so that:

$$Q = A\epsilon_0 E$$

$$Q = \frac{A\epsilon_0 V}{d}$$

Finally:

$$\mathbf{F} = Q\mathbf{E}_{\text{neg plate}}$$

$$\mathbf{F} = -\frac{A\epsilon_0 V^2}{2d^2} \hat{\mathbf{z}}$$

Parallel Cylinders:

The electric field due to cylinder 1 was found to be:

$$\mathbf{E}_1 = \frac{\lambda}{2\pi r\epsilon_0} \hat{\mathbf{r}}$$

Let us restrict ourselves to points along the x axis.

$$\mathbf{E}_1 = \frac{\lambda}{2\pi x\epsilon_0} \hat{\mathbf{x}}$$

The field due to cylinder 2 is

$$\mathbf{E}_2 = \frac{-\lambda}{2\pi(x-d)\epsilon_0} \hat{\mathbf{x}}$$

The total field is:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0} \left[\frac{1}{x} - \frac{1}{x-d} \right] \hat{\mathbf{x}}$$

The potential difference between the two cylinders can be found by integrating:

$$V = - \int_{a_1}^{d-a_2} \mathbf{E} \cdot d\mathbf{x}$$

$$V = \frac{-\lambda}{2\pi\epsilon_0} \int_{a_1}^{d-a_2} \left[\frac{1}{x} - \frac{1}{x-d} \right] dx$$

$$V = \frac{-\lambda}{2\pi\epsilon_0} \ln \left[\frac{(d-a_2)(d-a_1)}{a_1 a_2} \right]$$

With $d \gg a$, we can simplify this

$$V = \frac{-\lambda}{\pi\epsilon_0} \ln \left[\frac{d}{\sqrt{a_1 a_2}} \right]$$

Solve for the line charge density:

$$\lambda = \frac{-\pi\epsilon_0 V}{\ln \left[\frac{d}{\sqrt{a_1 a_2}} \right]}$$

$$\frac{\mathbf{F}}{l} = -\lambda \mathbf{E}_1(d, 0)$$

$$\boxed{\frac{\mathbf{F}}{l} = \frac{\pi\epsilon_0 V^2}{2d \left[\ln \left(\frac{d}{\sqrt{a_1 a_2}} \right) \right]^2} (-\hat{\mathbf{x}})}$$