Jackson 1.7 Homework Problem Solution
Dr. Christopher S. Baird
University of Massachusetts Lowell

C. S. BAIRD

UMASS
LOWELL

PROBLEM:

Two long, cylindrical conductors of radii a_1 and a_2 are parallel and separated by a distance d, which is large compared with either radius. Show that the capacitance per unit length is given approximately by

$$C \approx \pi \epsilon_0 \left(\ln \frac{d}{a} \right)^{-1}$$

where a is the geometrical mean of the two radii. Approximately what gauge wire (state diameter in millimeters) would be necessary to make a two-wire transmission line with a capacitance of 1.2×10^{-11} F/m if the separation of the wires was 0.5 cm? 1.5 cm? 5.0 cm?

SOLUTION:

Place the center of cylinder 1 at the origin and the center of cylinder 2 at $x = d$. Place a total charge per unit length $-Q/L$ on cylinder 1 and $+Q/L$ on cylinder 2. We first pretend cylinder 2 is not present and solve by superposition. Without cylinder 2, the field lines from cylinder 1 are radially outwards. This symmetry allows us to use Gauss's law in integral form. Draw a cylindrical Gaussian surface centered on cylinder 1 and enclosing it.

$$\oint \mathbf{E}_1 \cdot \mathbf{n} \, da = \frac{q_{enc}}{\epsilon_0}$$

$$\mathbf{E}_1 = \frac{-Q}{2 \pi r L \epsilon_0} \hat{\mathbf{r}}$$

The second cylinder by itself will create a similar field, only shifted to the right:

$$\mathbf{E}_2 = \frac{Q}{2 \pi |\mathbf{x} - d\,\hat{\mathbf{i}}|^2 L \epsilon_0} (\mathbf{x} - d\,\hat{\mathbf{i}})$$

So that the total field is:

$$\mathbf{E} = \frac{-Q}{2 \pi (x^2 + y^2) L \epsilon_0} (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) + \frac{Q}{2 \pi ((x-d)^2 + y^2) L \epsilon_0} ((x-d)\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}})$$

The potential is then:

$$\nabla \Phi = \frac{Q}{2 \pi (x^2 + y^2) L \epsilon_0} (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) - \frac{Q}{2 \pi ((x-d)^2 + y^2) L \epsilon_0} ((x-d)\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}})$$

To calculate conductance, we don't need to know the potential everywhere, only the potential difference between the two conductors. Also, the conductor surfaces are equipotentials, so we only need to know the potential at one point on the surface to know it everywhere. Let us calculate the potential difference between the points $(x = a_1, y = 0)$ and $(x = d - a_2, y = 0)$. Dot the above equation with $\hat{\mathbf{i}}$.

$$\frac{d\,\Phi}{d\,x} = \frac{Q}{2\pi(x^2 + y^2)L\,\epsilon_0}\,x - \frac{Q}{2\pi((x-d)^2 + y^2)L\,\epsilon_0}(x-d)$$

Integrate between our two points:

$$d\,\Phi = \frac{Q}{2\pi(x^2 + y^2)L\,\epsilon_0}\,x - \frac{Q}{2\pi((x-d)^2 + y^2)L\,\epsilon_0}(x-d)\,d\,x$$

$$\Delta V = \int_{a_1}^{d-a_2} d\,\Phi = \int_{a_1}^{d-a_2}\left[\frac{Q}{2\pi x\,L\,\epsilon_0} - \frac{Q}{2\pi(x-d)L\,\epsilon_0}\,d\,x\right]$$

$$\Delta V = \frac{Q}{2\pi\,\epsilon_0 L}\left[\int_{a_1}^{d-a_2}\frac{1}{x}\,d\,x - \int_{a_1}^{d-a_2}\frac{1}{x-d}\,d\,x\right]$$

$$\Delta V = \frac{Q}{2\pi\,\epsilon_0 L}\left[\ln\left(\frac{d-a_2}{a_1}\right) - \ln\left(\frac{-a_2}{a_1-d}\right)\right]$$

$$\Delta V = \frac{Q}{2\pi\,\epsilon_0 L}\left[\ln\left(\left(\frac{d-a_1}{a_2}\right)\left(\frac{d-a_2}{a_1}\right)\right)\right]$$

$$C\,L = \frac{Q}{\Delta V}$$

$$C = 2\pi\,\epsilon_0\left[\ln\left(\left(\frac{d-a_1}{a_2}\right)\left(\frac{d-a_2}{a_1}\right)\right)\right]^{-1}$$

For $d \gg a_1, a_2$

$$C \approx \pi\,\epsilon_0\left[\ln\left(\frac{d}{\sqrt{a_1 a_2}}\right)\right]^{-1}$$

$$\boxed{C \approx \pi\,\epsilon_0\left(\ln\frac{d}{a}\right)^{-1}} \quad \text{where} \quad a = \sqrt{a_1 a_2}$$

$$a \approx d\,e^{\frac{-\pi\,\epsilon_0}{C}}$$

For a two-wire transmission line with a capacitance of 1.2 x 10⁻¹¹ F/m, this becomes:

$$a \approx (0.1)d$$

or in terms of the diameter:

$$diameter \approx (0.2)d$$

If the separation of the wires was 0.5 cm, 1.5 cm, or 5.0 cm, the gauge should be:

1 mm, 3 mm, or 10 mm