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Jackson 1.6 Homework Problem Solution

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PROBLEM:

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of

- (a) two large, flat, conducting sheets of area A , separated by a small distance d ;
- (b) two concentric sphere with radii a, b ($b > a$)
- (c) two concentric conducting cylinders of length L , large compared to their radii a, b ($b > a$).
- (d) What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m?

SOLUTION:

In each case, the dimensions are such that we can neglect fringe effects and therefore can take advantage of symmetry.

(a) Parallel Plates

If we place a charge of $-Q$ on a plate at $z = 0$ and a charge of $+Q$ on a plate at $z = d$, then a constant voltage difference V is established between the two plates. The relationship between potential and electric field is:

$$\mathbf{E} = -\nabla \Phi$$

For a one-dimensional system (as this one is because the sheets are large enough to be approximated as infinite), the magnitude reduces to:

$$|\mathbf{E}| = \frac{d\Phi}{dz}$$

For a uniform electric field (as this must be because of the symmetry) this becomes:

$$E = \frac{\Phi}{z}$$

The potential difference between the two plates is then:

$$E = \frac{V}{d}$$

If we draw a Gaussian pillbox around a section of the positive plate, we will contain part of its charge. Apply Gauss's law in integral form over the surface of the Gaussian pillbox:

$$\oint \mathbf{E} \cdot \mathbf{n} da = \frac{q_{\text{enc}}}{\epsilon_0}$$

If we shrink the box properly, the part of the integral due to the sides of the pillbox surface become negligible, and we are left with the top and bottom of the pillbox. If we shrink the box small enough, the electric field becomes constant across the surface and can be taken out of the integral, so that:

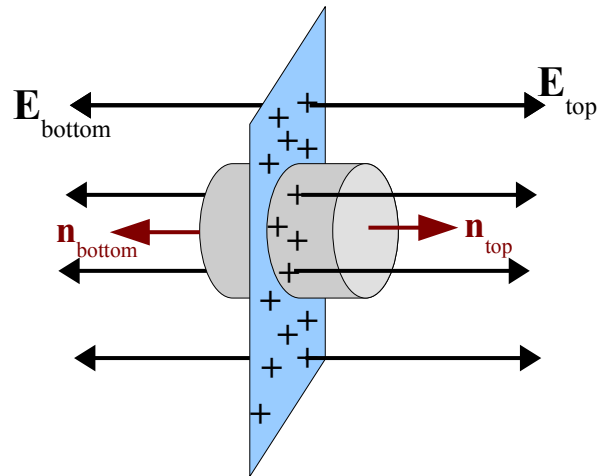
$$\mathbf{E}_{\text{top}} \cdot \mathbf{n}_{\text{top}} \int da + \mathbf{E}_{\text{bottom}} \cdot \mathbf{n}_{\text{bottom}} \int da = \frac{q_{\text{enc}}}{\epsilon_0}$$

In this particular case, the fields and normals point in the same direction (see diagram) and thus reduce down to a simple scalar product.

$$E_{\text{pos plate}} = \frac{q_{\text{enc}}}{2 A \epsilon_0}$$

$$E_{\text{pos plate}} = \frac{\sigma}{2 \epsilon_0}$$

$$E_{\text{pos plate}} = \frac{Q}{2 A \epsilon_0}$$



This is the field due to the positive plate. It should be obvious from the geometry that the field magnitude due to the negative plate is the same so that the total field is:

$$E = \frac{Q}{A \epsilon_0}$$

Substitute in the relationship between electric field and potential difference V found above and solve for Q/V which is capacitance:

$$\frac{Q}{V} = \frac{A \epsilon_0}{d}$$

$$C = \frac{\epsilon_0 A}{d}$$

(b) Concentric Spheres:

Place a charge $+Q$ on the sphere at $r = a$ and $-Q$ on the sphere at $r = b$. If we draw a spherical Gaussian surface around the inner sphere such that $a < r < b$, apply Gauss's law in integral form, and use symmetry, it is easy to show that between the spheres the electric field is:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$$

The electric potential is than

$$-\nabla\Phi = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$$

Dot both sides by the radial unit vector and integrate with respect to r to get the potential difference between the spheres:

$$\frac{d\Phi}{dr} = -\frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

$$\int_b^a d\Phi = -\int_b^a \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} dr$$

$$V = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

$$\frac{Q}{V} = \frac{4\pi\epsilon_0}{\left(\frac{1}{a} - \frac{1}{b} \right)}$$

$$C = \frac{4\pi\epsilon_0 ab}{b-a}$$

(c) Concentric Cylinders:

Place a charge $+Q$ on the cylinder at $r = a$ and $-Q$ on the cylinder at $r = b$. If we draw a cylindrical Gaussian surface around the inner cylinder such that $a < r < b$, apply Gauss's law in integral form, and use symmetry, it is easy to show that between the cylinders the electric field is:

$$\mathbf{E} = \frac{Q}{2\pi r\epsilon_0 L} \hat{\mathbf{r}}$$

Now find the potential difference:

$$d\Phi = -\frac{Q}{2\pi r\epsilon_0 L} dr$$

$$V = -\int_b^a \frac{Q}{2\pi r \epsilon_0 L} dr$$

$$V = \frac{Q}{2\pi \epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

$$\frac{Q}{V} = \frac{2\pi \epsilon_0 L}{\ln(b/a)}$$

$$\boxed{C = \frac{2\pi \epsilon_0 L}{\ln(b/a)}}$$

(d) Solve for b :

$$b = a e^{2\pi \epsilon_0 L/C}$$

$$b = (1 \text{ mm}) e^{2\pi (0.89 \times 10^{-11} \text{ F/m}) / (3 \times 10^{-11} \text{ F/m})}$$

$$b = 6 \text{ mm}$$

Also:

$$b = (1 \text{ mm}) e^{2\pi (8.9 \times 10^{-12} \text{ F/m}) / (3 \times 10^{-12} \text{ F/m})}$$

$$b = 10^8 \text{ mm or } 100 \text{ km}$$