Jackson 1.5 Homework Problem Solution

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PROBLEM:

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

SOLUTION:

The Poisson equation links charge densities and the electric scalar potential that they create. We use it here to find the charge density. We must perform a straight-forward differentiation in spherical coordinates.

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

Expand this in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{\rho}{\epsilon_0}$$

The potential is spherically symmetric, so that the potential depends only on the radial coordinate - the partial derivatives of the potential are all zero, except for the one with respect to the radial component.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)=-\frac{\rho}{\epsilon_0}$$

Evaluate the equation explicitly:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{1}{2}\alpha r\right)\right]\right)=-\frac{\rho}{\epsilon_0}$$

$$\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{e^{-\alpha r}}{r}\right]+r^2\frac{\partial}{\partial r}\left[\frac{\alpha}{2}e^{-\alpha r}\right]\right)=-\frac{\rho}{\epsilon_0}$$

$$\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\frac{1}{r}\frac{\partial}{\partial r}e^{-\alpha r}+e^{-\alpha r}\frac{\partial}{\partial r}\frac{1}{r}\right]-\frac{\alpha^2}{2}r^2 e^{-\alpha r}\right)=-\frac{\rho}{\epsilon_0}$$

$$\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(-\alpha r e^{-\alpha r}+r^2 e^{-\alpha r}\frac{\partial}{\partial r}\frac{1}{r}-\frac{\alpha^2}{2}r^2 e^{-\alpha r}\right)=-\frac{\rho}{\epsilon_0}$$

$$\frac{q}{4\pi\epsilon_0}\left(\frac{-\alpha}{r^2}e^{-\alpha r}+\frac{\alpha^3}{2}e^{-\alpha r}\right)+\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 e^{-\alpha r}\frac{\partial}{\partial r}\frac{1}{r}\right)=-\frac{\rho}{\epsilon_0}$$

$$\rho=\frac{-q\alpha^3}{8\pi}e^{-\alpha r}-e^{-\alpha r}\frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right)$$

Now we must be careful because $1/r$ blows up at the origin. Split the last term into two cases:

$$\rho=\frac{-q\alpha^3}{8\pi}e^{-\alpha r}-e^{-\alpha r}\frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right)\quad\text{if } r\approx 0$$

$$\rho=\frac{-q\alpha^3}{8\pi}e^{-\alpha r}-e^{-\alpha r}\frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right)\quad\text{if } r>0$$

Away from the origin, $1/r$ does not blow up and the derivatives can be evaluated normally. The last term ends up equating to zero, so that our equations now becomes:

$$\rho=\frac{-q\alpha^3}{8\pi}e^{-\alpha r}-e^{-\alpha r}\frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right)\quad\text{if } r\approx 0$$

$$\rho=\frac{-q\alpha^3}{8\pi}e^{-\alpha r}\quad\text{if } r>0$$

At $r\approx 0$, we have $e^{-\alpha r}=1$ so that the two cases become:

$$\rho=\frac{-q\alpha^3}{8\pi}-\frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right)\quad\text{if } r\approx 0$$

$$\rho=\frac{-q\alpha^3}{8\pi}e^{-\alpha r}\quad\text{if } r>0$$

Now use the relation:

$$\nabla^2\left(\frac{1}{r}\right)=-4\pi\delta(r)$$

which when evaluated explicitly becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)=-4\pi\delta(r)$$

Plug this into the above set of equations:

$$\rho=-\frac{q\alpha^3}{8\pi}+q\delta(r) \quad \text{if } r\approx 0$$

$$\rho=-\frac{q\alpha^3}{8\pi}e^{-\alpha r} \quad \text{if } r>0$$

Because the delta function is zero everywhere except at the origin, and because the first term of the first equation is just the specific $r = 0$ form of the first term of the second equation, the two cases can be combined into one case:

$$\boxed{\rho=-\frac{q\alpha^3}{8\pi}e^{-\alpha r}+q\delta(r)} \quad \text{for all } r$$

This corresponds physically to a positive point charge at the origin with one unit of elementary charge, and a finite cloud of negative charge that decays exponentially, but contains a total charge of one unit of elementary charge.

From a time-averaged perspective then, hydrogen in the ground state contains a positive point charge at the center and a circular cloud of negative charge. This is of course only useful for conceptualization purposes, because at atomic sizes the system behaves quantum mechanically, not classically.