Jackson 1.4 Homework Problem Solution

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PROBLEM:

Each of three charged spheres of radius a, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n ($n > -3$), has a total charge Q. Use Gauss's theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2$, $n = +2$.

SOLUTION:

a) The first sphere is conducting. As shown in Problem 1.1, the electric field inside the sphere is zero. To obtain the field outside the field, draw an integration sphere concentric to the conducting sphere and with some radius r. As shown in Problem 1.4, the electric field is normal to the conducting sphere, is thus normal to the integration surface, and is thus parallel to the integration surface's normal. Gauss's law becomes:

$$\oint_S E\, da = \frac{Q}{\epsilon_0}$$

Due to the symmetry of the conducting sphere and integration sphere, the electric field is constant over the integration surface and can be removed from the integral:

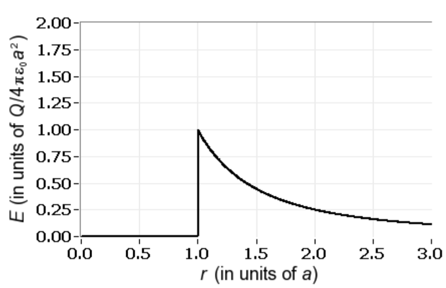
$$E \oint_S da = \frac{Q}{\epsilon_0}$$

The surface integral evaluates to the total area of the integration sphere with radius r.

$$E\, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

After rearranging:

$$\boxed{E = \frac{Q}{4\pi \epsilon_0 r^2}}$$



It should be noted that according to Coulomb's law, this is the same field as that created by a point charge Q at the origin. The sphere acts as if all of its charge where concentrated at its center even though it is actually spread uniformly over its surface.

b) The second sphere has a uniform charge density. Outside the sphere, the surface integration used in Gauss's law contains the same amount of charge as for the conducting sphere above and thus has the same solution:

$$\boxed{E=\frac{Q}{4\pi\epsilon_0 r^2}}$$

Inside the sphere there is a uniform charge density:

$$\rho=\frac{Q}{\frac{4}{3}\pi a^3}$$

Draw a spherical integration surface concentric with and inside the charged sphere, with a radius *r*. The total charge contained in this integration sphere is:

$$q=\left(\frac{Q}{\frac{4}{3}\pi a^3}\right)\left(\frac{4}{3}\pi r^3\right)$$

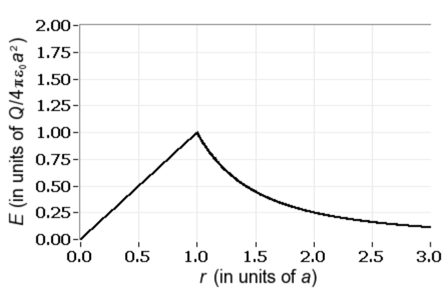
$$q=Q\frac{r^3}{a^3}$$

Due to parallel vectors and symmetry, as shown above, Gauss's law becomes:

$$E\oint_S da=\frac{q}{\epsilon_0}$$

$$E\,4\pi r^2=\frac{1}{\epsilon_0}Q\frac{r^3}{a^3}$$

$$\boxed{E=\frac{Qr}{4\pi\epsilon_0 a^3}}$$



c) The third sphere has a spherically symmetric charge density that varies radially as r^n. Outside the sphere, the surface integration used in Gauss's law contains the same amount of charge as for the conducting sphere above and thus has the same solution:

$$\boxed{E=\frac{Q}{4\pi\epsilon_0 r^2}}$$

Inside the sphere we must first find the full form of the charge density. The charge density has the form:

$$\rho = A r^n$$

To determine the constant *A*, we integrate the charge density over the whole sphere and set it equal to the total charge:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} A r^n r^2 \, dr \sin\theta \, d\theta \, d\phi = Q$$

$$A = \frac{Q}{4\pi \int_0^a r^{n+2} dr}$$

$$A = \frac{Q}{4\pi\left[\frac{a^{n+3}}{n+3}\right]}$$

The charge density now has the form:

$$\rho = \frac{Q}{4\pi\left[\frac{a^{n+3}}{n+3}\right]} r^n$$

Draw a spherical integration surface concentric with and inside the charged sphere, with a radius *r*. The total charge contained in this integration sphere is obtained by integrating:

$$q = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho(r') r'^2 \, dr' \sin\theta \, d\theta \, d\phi$$

$$q = 4\pi \int_0^r \rho(r') r'^2 \, dr'$$

$$q = 4\pi \int_0^r \frac{Q}{4\pi\left[\frac{a^{n+3}}{n+3}\right]} r'^n r'^2 \, dr'$$

$$q = 4\pi \frac{Q}{4\pi\left[\frac{a^{n+3}}{n+3}\right]} \int_0^r r'^{n+2} \, dr'$$

$$q = 4\pi \frac{Q}{4\pi\left[\frac{a^{n+3}}{n+3}\right]} \frac{r^{n+3}}{n+3}$$

$$q = Q\frac{r^{n+3}}{a^{n+3}}$$

Due to parallel vectors and symmetry, as shown above, Gauss's law becomes:

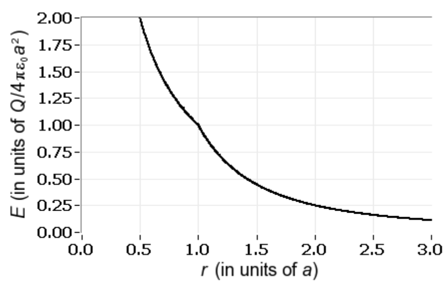
$$E \oint_S da = \frac{q}{\epsilon_0}$$

$$E\, 4\pi r^2 = \frac{1}{\epsilon_0} Q \frac{r^{n+3}}{a^{n+3}}$$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}\left(\frac{r^{n+3}}{a^{n+3}}\right)}$$

For n = -2:

$$\boxed{E = \frac{Q}{4\pi\epsilon_0 r\, a}}$$



For n = 2:

$$\boxed{E = \frac{Q r^3}{4\pi\epsilon_0 a^5}}$$