Jackson 1.3 Homework Problem Solution

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PROBLEM:

Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

(d) The same as part (c), but using spherical coordinates.

SOLUTION:

The easiest method to use is to set a Dirac delta for every dimension that has an infinitely thin appearance. Multiply this by some arbitrary parameter, integrate over the whole object, set this equal to the total charge, then solve for the arbitrary parameter.

(a) For the spherical shell, the charge distribution is only thin in the radial direction.

$$\rho(r,\theta,\phi)=A\delta(r-R)$$

Now integrate over all space and set it equal to the total charge Q.

$$Q=\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\rho(r,\theta,\phi)r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$Q=4\pi A\int_0^{\infty}\delta(r-R)r^2\, dr$$

$$Q=4\pi R^2 A$$

$$A=\frac{Q}{4\pi R^2}$$

$$\boxed{\rho(r,\theta,\phi)=\frac{Q}{4\pi R^2}\delta(r-R)}$$

This answer should be obvious now. It is just the total charge divided by the area of a sphere times the delta.

(b) For the cylindrical surface:

$$\rho(r,\phi,z)=A\delta(r-b)$$

$$\lambda=\int_0^{2\pi}\int_0^{\infty}\rho(r,\phi,z)r\,dr\,d\theta$$

$$\lambda=A\int_0^{2\pi}d\theta\int_0^{\infty}\delta(r-b)r\,dr$$

$$\lambda=A2\pi b$$

$$A=\frac{\lambda}{2\pi b}$$

$$\boxed{\rho(r,\phi,z)=\frac{\lambda}{2\pi b}\delta(r-b)}$$

Again, this should be obvious that this is the surface charge density time the delta, where the surface charge density is the linear charge density divided by the circumference of the cylinder.

(c) For the flat disc, we must use the step function *H* in the radial direction.

$$\rho(r,\phi,z)=A\delta(z)H(R-r)$$

$$Q=\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\rho(r,\phi,z)r\,dr\,d\theta\,dz$$

$$Q=A\int_{-\infty}^{\infty}\delta(z)dz\int_0^{2\pi}d\theta\int_0^{\infty}H(R-r)r\,dr$$

$$Q=A2\pi\int_0^{R}r\,dr$$

$$A=\frac{Q}{\pi R^2}$$

$$\boxed{\rho(r,\phi,z)=\frac{Q}{\pi R^2}\delta(z)H(R-r)}$$

Again, it should be obvious that this is the deltas times the surface charge density, which is the total charge divided by the area of the disc.

(d) For the flat disc in spherical coordinates try:

$$\rho(r,\theta,\phi)=A\frac{\delta(\theta-\pi/2)}{r}H(R-r)$$

$$Q=\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\rho(r,\theta,\phi)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$Q=A\int_0^{2\pi}d\phi\int_0^{\pi}\delta(\theta-\pi/2)\sin\theta\,d\theta\int_0^{\infty}H(R-r)r\,dr$$

$$A=\frac{Q}{\pi R^2}$$

$$\boxed{\rho(r,\theta,\phi)=\frac{Q}{\pi R^2}\frac{\delta(\theta-\pi/2)}{r}H(R-r)}$$