Jackson 1.2 Homework Problem Solution

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PROBLEM:

The Dirac delta function in three dimensions can be taken as the improper limit as $\alpha \to 0$ of the Gaussian function

$$D(\alpha; x, y, z) = (2\pi)^{-3/2} \alpha^{-3} \exp\left[-\frac{1}{2\alpha^2}(x^2+y^2+z^2)\right]$$

Consider a general orthogonal coordinate system specified by the surfaces u = constant, v = constant, w = constant, with length elements du/U, dv/V, dw/W in the three perpendicular directions. Show that

$$\delta(\mathbf{x}-\mathbf{x}') = \delta(u-u')\delta(v-v')\delta(w-w') \cdot UVW$$

by considering the limit of the Gaussian above. Note that as $\alpha \to 0$ only the infinitesimal length element need be used for the distance between the points in the exponent.

SOLUTION:

Start with the general property of a Dirac delta:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,dx\,dy\,dz = 1$$

Substitute in our representation:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lim_{\alpha\to 0} D(\alpha; x, y, z)\,dx\,dy\,dz = 1$$

Now transform the volume element into the new coordinate system

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lim_{\alpha\to 0} D(\alpha; x, y, z)\frac{du}{U}\frac{dv}{V}\frac{dw}{W} = 1$$

We do not know exactly how the one system of coordinates transforms into the other, so we cannot transform D in a direct manner. Let us instead define an intermediate variable function F according to:

$$F(u, v, w) = \lim_{\alpha\to 0} D(\alpha; x, y, z \to u, v, w)\frac{1}{U\,V\,W}$$

With this definition our integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v, w)\,du\,dv\,dw = 1$$

Because we are integrating over all space, we are free to make a change of variables which just shifts the origin.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u-u',v-v',w-w')\,du\,dv\,dw=1$$

Comparing this to the very first equation we see that it is identical except with different integration labels and therefore:

$$F(u-u',v-v',w-w')=\delta(u-u')\,\delta(v-v')\,\delta(w-w')$$

so that, after plugging back in, we have

$$\delta(u-u')\,\delta(v-v')\,\delta(w-w')=\lim_{\alpha\to 0} D(\alpha;x,y,z\to u,v,w)\frac{1}{U\,V\,W}$$

Solve for *D*:

$$\lim_{\alpha\to 0} D(\alpha;x,y,z\to u,v,w)=\delta(u-u')\,\delta(v-v')\,\delta(w-w')\,U\,V\,W$$

The entity on the left, no matter what coordinate system it is represented in, is just what we mean by the general three-dimensional Dirac delta:

$$\boxed{\delta(\mathbf{x}-\mathbf{x}')=\delta(u-u')\,\delta(v-v')\,\delta(w-w')\cdot UVW}$$

Now note that we never used the explicit form of *D*, so we have solved the problem in a way that the book did not intend. Let us try solving the problem using a method that uses the explicit form of *D*.

$$D(\alpha;x,y,z)=(2\pi)^{-3/2}\alpha^{-3}\exp\left[-\frac{1}{2\alpha^2}(x^2+y^2+z^2)\right]$$

Make a change of variables $x\to x-x'$, etc. (Otherwise we will not end up with the most general case.)

$$D(\alpha;x-x',y-y',z-z')=(2\pi)^{-3/2}\alpha^{-3}\exp\left[-\frac{1}{2\alpha^2}((x-x')^2+(y-y')^2+(z-z')^2)\right]$$

As $\alpha\to 0$, *D* will become zero unless $x-x'$ approaches zero as well. In calculus, we remember that $x-x'$ approaching zero becomes dx. Therefore we have:

$$D(\alpha;x-x',y-y',z-z')=(2\pi)^{-3/2}\alpha^{-3}\exp\left[-\frac{1}{2\alpha^2}((dx)^2+(dy)^2+(dz)^2)\right]$$

We recognize the last part in parentheses as the incremental arc length element *ds* squared:

$$D(\alpha;x-x',y-y',z-z')=(2\pi)^{-3/2}\alpha^{-3}\exp\left[-\frac{1}{2\alpha^2}ds^2\right]$$

Expand the arc length in the new coordinate system:

$$D=(2\pi)^{-3/2}\alpha^{-3}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{du^2}{U^2}+\frac{dv^2}{V^2}+\frac{dw^2}{W^2}\right)\right]$$

Note that $dx \neq du/U$ and we are not making that claim here. Rather, the entire three-dimensional incremental arc length *ds* is the same in all orthogonal coordinate systems. Now expand the increments back into differences:

$$D=(2\pi)^{-3/2}\alpha^{-3}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{(u-u')^2}{U^2}+\frac{(v-v')^2}{V^2}+\frac{(w-w')^2}{W^2}\right)\right]$$

$$D=\left[\frac{e^{-(u-u')^2/2\alpha^2U^2}}{\sqrt{2\pi}\,\alpha}\right]\left[\frac{e^{-(v-v')^2/2\alpha^2V^2}}{\sqrt{2\pi}\,\alpha}\right]\left[\frac{e^{-(w-w')^2/2\alpha^2W^2}}{\sqrt{2\pi}\,\alpha}\right]$$

Now make the substitution $\alpha \rightarrow \alpha_1/U$ in the first bracket, $\alpha \rightarrow \alpha_2/V$ in the second bracket, and $\alpha \rightarrow \alpha_3/W$ in the last bracket. We can do this as long as we let α_1, α_2, and α_3 go to zero just like we were letting α go to zero.

$$D=\left[\frac{e^{-(u-u')^2/2\alpha_1^2}}{\sqrt{2\pi}\,\alpha_1}\right]\left[\frac{e^{-(v-v')^2/2\alpha_2^2}}{\sqrt{2\pi}\,\alpha_2}\right]\left[\frac{e^{-(w-w')^2/2\alpha_3^2}}{\sqrt{2\pi}\,\alpha_3}\right]U\,V\,W$$

We now let the alpha's approach zero. Each term in brackets on the right side becomes a one-dimensional linear Dirac delta. The left side becomes the general expression for the three-dimensional Dirac delta:

$$\boxed{\delta(\mathbf{x}-\mathbf{x}')=\delta(u-u')\,\delta(v-v')\,\delta(w-w')\cdot UVW}$$

Now this is a very useful result. Suppose we have a point charge. In spherical coordinates, we can find the representation of its Dirac delta using the above expression. For spherical coordinates $u=r,\ v=\theta,\ w=\phi$ and the length elements are $dr,\ r\,d\theta,\ r\sin\theta\,d\phi$ so that $U=1,\ V=\frac{1}{r},\ W=\frac{1}{r\sin\theta}$ and

$$\delta(\mathbf{x}-\mathbf{x}')=\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')\frac{1}{r^2\sin\theta} \quad \text{(Spherical Coordinates)}$$

Note that it is fairly straight-forward to prove using Dirac delta properties that $\delta(\theta-\theta')/\sin\theta=\delta(\cos\theta-\cos\theta')$ so that the three-dimensional Dirac delta in spherical coordinates is often written

$$\delta(\mathbf{x}-\mathbf{x}')=\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')\frac{1}{r^2}$$

as it is on p. 120 in *Jackson*.

Similarly in cylindrical coordinates, $u=r,\ v=\theta,\ w=z$ and the length elements are $dr,\ r\,d\theta,\ dz$ so that

$U=1, V=\frac{1}{r}, W=1$ and

$$\delta(\mathbf{x}-\mathbf{x}')=\delta(r-r')\delta(\theta-\theta')\delta(z-z')\frac{1}{r} \quad \text{(Cylindrical Coordinates)}$$

Assume that instead of a point charge, we have a line charge shaped into a ring, centered on the *z* axis, located at some radius *r*' and polar angle *θ*'. The charge is distributed along the ring according to the line charge density $\lambda(\phi)$. The total charge density in this case would be:

$$\rho=\delta(u-u')\delta(v-v')U\,V\,\lambda(w) \quad \text{(Spherical Coordinates)}$$

$$\rho=\frac{\delta(r-r')\delta(\theta-\theta')\lambda(\phi)}{r} \quad \text{(Spherical Coordinates)}$$