PROBLEM:
The Dirac delta function in three dimensions can be taken as the improper limit as $\alpha \to 0$ of the Gaussian function

$$D(\alpha; x, y, z) = (2\pi)^{-3/2} \alpha^{-3} \exp \left[ -\frac{1}{2\alpha^2} (x^2 + y^2 + z^2) \right]$$

Consider a general orthogonal coordinate system specified by the surfaces $u = \text{constant}$, $v = \text{constant}$, $w = \text{constant}$, with length elements $du/U$, $dv/V$, $dw/W$ in the three perpendicular directions. Show that

$$\delta (x - x') = \delta (u - u') \delta (v - v') \delta (w - w') \cdot UVW$$

by considering the limit of the Gaussian above. Note that as $\alpha \to 0$ only the infinitesimal length element need be used for the distance between the points in the exponent.

SOLUTION:
Start with the general property of a Dirac delta:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(y) \delta(z) dx dy dz = 1$$

Substitute in our representation:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lim_{\alpha \to 0} D(\alpha; x, y, z) dx dy dz = 1$$

Now transform the volume element into the new coordinate system

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lim_{\alpha \to 0} D(\alpha; x, y, z) \frac{du}{U} \frac{dv}{V} \frac{dw}{W} = 1$$

We do not know exactly how the one system of coordinates transforms into the other, so we cannot transform $D$ in a direct manner. Let us instead define an intermediate variable function $F$ according to:

$$F(u, v, w) = \lim_{\alpha \to 0} D(\alpha; x, y, z \to u, v, w) \frac{1}{UVW}$$

With this definition our integral becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v, w) du dv dw = 1$$
Because we are integrating over all space, we are free to make a change of variables which just shifts the origin.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u-u', v-v', w-w') \, du \, dv \, dw = 1
\]

Comparing this to the very first equation we see that it is identical except with different integration labels and therefore:

\[
F(u-u', v-v', w-w') = \delta(u-u') \delta(v-v') \delta(w-w')
\]

so that, after plugging back in, we have

\[
\delta(u-u') \delta(v-v') \delta(w-w') = \lim_{\alpha \to 0} D(\alpha; x, y, z \to u, v, w) \frac{1}{U V W}
\]

Solve for \(D\):

\[
\lim_{\alpha \to 0} D(\alpha; x, y, z \to u, v, w) = \delta(u-u') \delta(v-v') \delta(w-w') U V W
\]

The entity on the left, no matter what coordinate system it is represented in, is just what we mean by the general three-dimensional Dirac delta:

\[
\delta(x-x') = \delta(u-u') \delta(v-v') \delta(w-w') U V W
\]

Now note that we never used the explicit form of \(D\), so we have solved the problem in a way that the book did not intend. Let us try solving the problem using a method that uses the explicit form of \(D\).

\[
D(\alpha; x, y, z) = (2\pi)^{-3/2} \alpha^{-3} \exp \left[ -\frac{1}{2\alpha^2} (x^2 + y^2 + z^2) \right]
\]

Make a change of variables \(x \to x-x', \) etc. (Otherwise we will not end up with the most general case.)

\[
D(\alpha; x-x', y-y', z-z') = (2\pi)^{-3/2} \alpha^{-3} \exp \left[ -\frac{1}{2\alpha^2} ((x-x')^2 + (y-y')^2 + (z-z')^2) \right]
\]

As \(\alpha \to 0\), \(D\) will become zero unless \(x-x'\) approaches zero as well. In calculus, we remember that \(x-x'\) approaching zero becomes \(dx\). Therefore we have:

\[
D(\alpha; x-x', y-y', z-z') = (2\pi)^{-3/2} \alpha^{-3} \exp \left[ -\frac{1}{2\alpha^2} ((dx)^2 + (dy)^2 + (dz)^2) \right]
\]

We recognize the last part in parentheses as the incremental arc length element \(ds\) squared:

\[
D(\alpha; x-x', y-y', z-z') = (2\pi)^{-3/2} \alpha^{-3} \exp \left[ -\frac{1}{2\alpha^2} ds^2 \right]
\]
Expand the arc length in the new coordinate system:

\[ D = (2\pi)^{-3/2} \alpha^{-3} \exp \left[ -\frac{1}{2\alpha^2} \left( \frac{du^2}{U^2} + \frac{dv^2}{V^2} + \frac{dw^2}{W^2} \right) \right] \]

Note that \( dx \neq du/U \) and we are not making that claim here. Rather, the entire three-dimensional incremental arc length \( ds \) is the same in all orthogonal coordinate systems. Now expand the increments back into differences:

\[ D = (2\pi)^{-3/2} \alpha^{-3} \exp \left[ -\frac{1}{2\alpha^2} \left( \frac{(u-u')^2}{U^2} + \frac{(v-v')^2}{V^2} + \frac{(w-w')^2}{W^2} \right) \right] \]

Now make the substitution \( \alpha \to \alpha_1/U \) in the first bracket, \( \alpha \to \alpha_2/V \) in the second bracket, and \( \alpha \to \alpha_3/W \) in the last bracket. We can do this as long as we let \( \alpha_1, \alpha_2, \alpha_3 \to 0 \) just like we were letting \( \alpha \to 0 \).

\[ D = \frac{e^{-(u-u')^2/2\alpha_1^2 U^2}}{\sqrt{2\pi \alpha_1}} \frac{e^{-(v-v')^2/2\alpha_2^2 V^2}}{\sqrt{2\pi \alpha_2}} \frac{e^{-(w-w')^2/2\alpha_3^2 W^2}}{\sqrt{2\pi \alpha_3}} U V W \]

We now let the alpha's approach zero. Each term in brackets on the right side becomes a one-dimensional linear Dirac delta. The left side becomes the general expression for the three-dimensional Dirac delta:

\[ \delta(x-x') = \delta(u-u') \delta(v-v') \delta(w-w') \cdot U V W \]

Now this is a very useful result. Suppose we have a point charge. In spherical coordinates, we can find the representation of its Dirac delta using the above expression. For spherical coordinates \( u=r, \nu=\theta, w=\phi \) and the length elements are \( dr, r \, d \theta, r \sin \theta \, d \phi \) so that \( U=1, V=\frac{1}{r}, W=\frac{1}{r \sin \theta} \) and

\[ \delta(x-x') = \delta(r-r') \delta(\theta-\theta') \delta(\phi-\phi') \frac{1}{r^2 \sin \theta} \] (Spherical Coordinates)

Note that it is fairly straight-forward to prove using Dirac delta properties that \( \delta(\theta-\theta')/\sin \theta = \delta(\cos \theta - \cos \theta') \) so that the three-dimensional Dirac delta in spherical coordinates is often written

\[ \delta(x-x') = \delta(r-r') \delta(\cos \theta - \cos \theta') \delta(\phi-\phi') \frac{1}{r^2} \]

as it is on p. 120 in *Jackson*.

Similarly in cylindrical coordinates, \( u=r, \nu=\theta, w=z \) and the length elements are \( dr, r \, d \theta, dz \) so that
\[ U = 1, V = \frac{1}{r}, W = 1 \] and
\[ \delta(x - x') = \delta(r - r') \delta(\theta - \theta') \delta(z - z') \frac{1}{r} \] (Cylindrical Coordinates)

Assume that instead of a point charge, we have a line charge shaped into a ring, centered on the z axis, located at some radius \( r' \) and polar angle \( \theta' \). The charge is distributed along the ring according to the line charge density \( \lambda(\phi) \). The total charge density in this case would be:

\[ \rho = \delta(u - u') \delta(v - v') U V \lambda(w) \] (Spherical Coordinates)

\[ \rho = \frac{\delta(r - r') \delta(\theta - \theta') \lambda(\phi)}{r} \] (Spherical Coordinates)