



C. S. BAIRD

Jackson 1.1 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell



UMASS
LOWELL

PROBLEM:

Use Gauss's theorem $\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{q}{\epsilon_0}$ and $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ to prove the following:

- Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)
- A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from fields due to charges placed inside it.
- The electric field at the surface of the conductor is normal to the surface and has a magnitude σ/ϵ_0 , where σ is the charge density per unit area on the surface.

SOLUTION:

a) First, the problem contains the unstated assumption that what is wanted is the location of the charges in *static equilibrium*. This assumption is valid because this problem is found in the chapter on electrostatics. Static equilibrium (the lack of movement) can only exist when *free* charges are present if there are no electric fields in the volume where the free charges exist. Otherwise, the charges, being free, would accelerate in response to the electric fields. Therefore, *the electric field inside a conductor is zero*. Note that this problem also assumes that a conductor is a perfect conductor, i.e. charges are perfectly free to move within the conductor. In real life, no conductor is perfect, so that all of the concepts derived in this problem are only approximations to real life materials.

At the surface of a conductor, the charges are not free in the direction normal to the conductor's surface (since they are bound inside the conducting material), and therefore the *only electric field that can exist in or on a conductor is an electric field at the surface of the conductor that is normal to the surface*. An outwardly normal electric field exists for positive surface charge and an inwardly normal electric field exists for negative surface charge density. This last part is true because negative charges feel a force $\mathbf{F} = -|q| \mathbf{E}$, so that if the net electric force on a negative charge is to be pointed outwards, the electric field must be pointed in the opposite direction. The component of the electric field at the surface of a conductor that is tangential to the surface must be zero, otherwise the charges would feel a force from this field and, being free in this direction, would move in response to the force, which contradicts our requirement of static equilibrium. So the tangential component must be zero. Thus far we have proven two concepts:

Concept 1. The electric field everywhere inside a conductor in static equilibrium is zero.

Concept 2. The electric field at the surface of a conductor is normal to the surface.

We can draw any arbitrary, closed Gaussian surface completely inside the conductor and it will always have zero electric field at every point on the Gaussian surface since the electric field is zero everywhere

inside the conductor. The integral of a bunch of zeroes is still zero, so that the integral of the electric field over any interior Gaussian surface is zero and so Gauss's law,

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{q}{\epsilon_0} \text{ becomes } 0 = q.$$

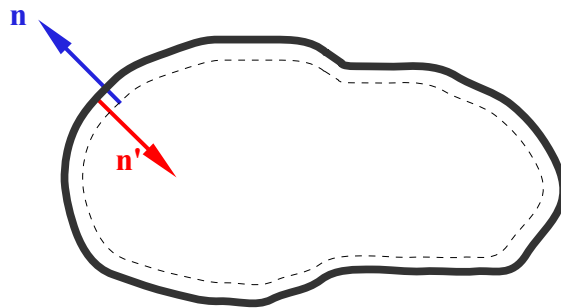
Because the closed Gaussian surface is arbitrary, it can be chosen to be very small and centered on any point in the conductor's interior, as long as the entire surface is inside the conductor, and it will always contain zero charge. Therefore, every point inside a conductor has zero charge. If there is any free electric charge at all in a conductor, it therefore must all end up on the conductor's surface when the conductor reaches static equilibrium. Physically, we can understand this as the net free charges mutually repelling each other and accelerating away from each other under this repulsion until they hit up against the conductor's surface. We have therefore proven:

Concept 3. All of the free electric charge in a conductor resides on its surface.

b) We now consider a closed, hollow, conducting shell. We have already shown that the electric field inside the material of the conducting shell itself has zero field and zero charge, but what about in the inner hollow region? Since the hollow region is hollow, it contains no charge by definition. Therefore, any Gaussian surface that is completely contained in the hollow region contains no charge and therefore has an integral that is zero according to Gauss's law:

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = 0 \text{ because } q = 0$$

Let us place a Gaussian surface right along the inner surface of the conducting shell. The unit vector \mathbf{n} in the above equation is outwardly normal to the Gaussian surface, and is therefore in the opposite direction of the normal to the conductor's inner surface, \mathbf{n}' .



Since the electric field on the surface of a conductor must be normal to the material's surface, the electric field on the inner surface of the conducting shell (if it exists) must also be normal to the Gaussian surface:

$\mathbf{E} \cdot \mathbf{n} = \pm |\mathbf{E}|$, so that:

$$\oint_S (\pm |\mathbf{E}|) da = 0$$

Note that the dot product can be positive or negative: positive if the electric field at the inner conducting surface points towards infinity, which would be the case for a negative surface charge

density pressed against the inner conducting surface; and negative if the electric field at the inner conducting surface points towards the hollow interior of the shell, which would be the case for a positive surface charge density pressed against the inner conducting surface. Also note that, in general, the electric field can point at certain locations on the Gaussian surface towards infinity and at other locations on the Gaussian surface towards the hollow region. Therefore, the sign of the dot product is not a constant, but is a function of location at this point in the derivation.

For simplicity, assume the conducting shell is grounded. This means that the conductor is connected to an effectively infinite source or sink of electric charge. When an external charge is placed near the conducting shell, its field draws charges in the shell towards it. When the shell is grounded, all of the charge it draws comes from the ground and becomes an excess charge on the shell. Therefore, the surface charge density everywhere on the shell has the same sign. For example, if the external charge is positive, the shell is everywhere negatively charged. If the external charge is negative, the shell is everywhere positively charged. If one end of the shell were positive and the other end negative, it would draw more charge from the ground until this went away.

Since the charge density and electric field on a conductor's surface are directly related (as we will prove in part c), the surface charge density having the same sign everywhere means that $\mathbf{E} \cdot \mathbf{n}$ has the same sign everywhere. In other words, the electric field on the inner shell's surface is everywhere pointing inwards or is everywhere pointing outwards. The sign of the dot product is therefore constant over the Gaussian surface and can come out of the integral:

$$\pm \oint_S |\mathbf{E}| da = 0$$

Note that this is only true if we have a solid closed conducting shell. If we took a sphere and cut it in half so that the two halves are insulated from each other, we could positively charge one half and negatively charge the other half and get inwards pointing electric fields on one half and outward electric fields on the other half.

The integrand in the equation above is now always positive or zero. Integrating over an integrand that is always non-zero and positive gives a positive number (since an integral is essentially a sum over little pieces and a sum over positive numbers is a positive number). The only way that the integral can be zero is if the integrand is everywhere zero. Therefore, the electric field is everywhere zero along the Gaussian surface. We have just proven:

Concept 4. The electric field on the inner surface of a closed, grounded conducting shell is zero.

We can choose a slightly different Gaussian surface that is still entirely in the hollow region, and has most of its surface on the shell's inner surface. The enclosed charge is still zero and the part of the integral along the shell's inner surface is zero (call this part of the surface A) because the electric field is zero here. Therefore, the integral over the remaining part of the Gaussian surface (call this surface B) must be zero:

$$\int_A |\mathbf{E}| da + \int_B |\mathbf{E}| da = 0 \text{ where the closed surface } S = \text{surface } A + \text{surface } B.$$

$$\int_B |\mathbf{E}| da = 0 \text{ since } \int_A |\mathbf{E}| da = 0$$

Since the Gaussian surface is arbitrary, as long as it is all in the hollow region, surface A can be so small that the electric field becomes effectively constant and can come out of the integral:

$$|\mathbf{E}| \int_A da = 0$$

$$|\mathbf{E}| = 0$$

And since the surface A can be drawn anywhere in the hollow region, the electric field must be zero everywhere in the hollow region. The shell therefore shields the hollow region from external fields and charges. We have shown:

Concept 5. The electric field everywhere inside the hollow region of a closed, grounded, conducting shell is zero, even if there are charges external to the shell.

Note that if the conductor is not grounded, but is insulated, we can treat the solution to this new problem as the solution to the grounded conductor problem plus the solution to a problem where the conductor is insulated but there is no external charge, because electric fields obey superposition. In other words, if an insulated conducting shell has an original charge Q , and the equivalently-shaped grounded conducting shell in the presence of an external charge has a total induced charge q , then the insulated shell will have a q portion of its total charge be effected by the external charge in the same way as a grounded shell, and the remaining $(Q - q)$ electric charge will not be effected by the external electric charge. Since this remaining portion is not affected by the external charge, it will spread out effectively uniformly over the shell, and the field in the hollow region due to this remaining charge will be zero since it is spread uniformly. The sum of an electric field of zero in the hollow region from the q portion plus an electric of zero in the hollow region from the $(Q - q)$ portion is still an electric field of zero. Therefore:

Concept 6. The electric field everywhere inside the hollow region of a closed, insulated, conducting shell is zero, even if there are charges external to the shell.

As a separate case, we place an isolated charge q inside the hollow region of an originally uncharged, insulated conducting shell and want to know the field outside the shell. We now place a Gaussian surface just outside the shell and enclosing it entirely. The total charge enclosed is now q , and so by Gauss's law, the fields outside the shell cannot be zero, but must depend on the inner charge q . We see that:

Concept 7. A closed, insulated, conducting shell does not shield the exterior from fields created by charges in its hollow region.

Although the problem did not ask for it, for completeness, let's look at the case of an isolated charge inside a closed, *grounded*, conducting shell. Since the shell is grounded, placing a charge q inside the hollow region draws a charge $-q$ from ground unto the shell. The total charge enclosed by a Gaussian surface surrounding everything is therefore $(q) + (-q) = 0$. Since the electric field is everywhere perpendicular to the outer conducting surface and everywhere the same sign, the electric field must be everywhere zero external to the shell by the same arguments as used above. Therefore:

Concept 8. A closed, grounded, conducting shell does shield the exterior from fields created by charges in its hollow region.

c) We draw a closed pillbox surface half-in and half-out of the surface of the conductor and integrate the electric field over the surface. There is no electric field tangential to the conductor's surface, thus the sides of the pillbox contribute nothing to the integral. Also, the electric field is zero inside the conductor, thus the bottom of the pillbox contributes nothing to the integral. All that is left is the top of the pillbox. The surface normal of its top and the electric field are parallel since the electric field is normal to the surface as we showed above, so that Gauss's law becomes:

$$\int_{\text{top}} E da = \frac{q}{\epsilon_0}$$

where q is the charge contained inside the pillbox. Now shrink the pillbox until it is very small. The electric field is effectively constant over a very small surface and can be taken out of the integral, so that the integral is evaluated to just be the total area of the pillbox top:

$$E A_{\text{top}} = \frac{q}{\epsilon_0}$$

Rearranging: $E = \frac{q}{A_{\text{top}}} \frac{1}{\epsilon_0}$

Define σ as the charge per unit area q/A_{top} and the equation becomes:

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$