Jackson 1.14 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell

PROBLEM:

Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface *S* bounding the volume *V*. Apply Green's theorem (1.35) with integration variables **y** and $\phi = G(\mathbf{x},\mathbf{y})$ and $\psi = G(\mathbf{x}',\mathbf{y})$, with $\nabla_y^2 G(\mathbf{z},\mathbf{y}) = -4\pi\,\delta(\mathbf{y}-\mathbf{z})$. Find an expression for the difference $[G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}', \mathbf{x})]$ in terms of an integral over the boundary surface *S*.

(a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\mathbf{x}, \mathbf{x}')$ must be symmetric in **x** and **x'**.

(b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(\mathbf{x}, \mathbf{x}')$ to show that $G_N(\mathbf{x}, \mathbf{x}')$ is not symmetric in general, but that $G_N(\mathbf{x}, \mathbf{x}') - F(\mathbf{x})$ is symmetric in **x** and **x'**, where

$$F(\mathbf{x}) = \frac{1}{S}\oint_S G_N(\mathbf{x},\mathbf{y})\,da_y$$

(c) Show that the addition of $F(\mathbf{x})$ to the Green function does not affect the potential $\Phi(\mathbf{x})$. See problem 2.36 for an example of the Neumann Green function.

SOLUTION:

The electrostatic Green function for Dirichlet and Neumann boundary conditions is:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G\,d^3\mathbf{x}' + \frac{1}{4\pi}\oint_S \left(G\frac{d\Phi}{dn'} - \Phi\frac{dG}{dn'}\right)da'$$

Green's theorem (1.35) is:

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]da$$

With integration variables **y** and $\phi = G(\mathbf{x},\mathbf{y})$ and $\psi = G(\mathbf{x}',\mathbf{y})$, and with $\nabla_y^2 G(\mathbf{z},\mathbf{y}) = -4\pi\,\delta(\mathbf{y}-\mathbf{z})$, this equation becomes:

$$-4\pi\int_V \left(G(\mathbf{x},\mathbf{y})\delta(\mathbf{y}-\mathbf{x}') - G(\mathbf{x}',\mathbf{y})\delta(\mathbf{y}-\mathbf{x})\right)d^3y = \oint_S \left[G(\mathbf{x},\mathbf{y})\frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right]da_y$$

$$\boxed{[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})] = -\frac{1}{4\pi}\oint_S \left[G(\mathbf{x},\mathbf{y})\frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right]da_y}$$

(a) For Dirichlet boundary conditions on the potential, Φ is known on the surface and F can be chosen to make $G_D = 0$ on the surface. The electrostatic Green function becomes:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}') G_D d^3\mathbf{x}' - \frac{1}{4\pi}\oint_S \Phi \frac{d G_D}{d n'} da'$$

The green function G_D in this case can be shown to be symmetric in **x** and **x'** by using the general form from above:

$$[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})] = -\frac{1}{4\pi}\oint_S \left[G(\mathbf{x},\mathbf{y}) \frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} \right] da_y$$

For Dirichlet boundary conditions, as stated above, $G_D = 0$ on the surface. The leads to:

$$[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})] = -\frac{1}{4\pi}\oint_S \left[(0)\frac{\partial(0)}{\partial n} - (0)\frac{\partial(0)}{\partial n} \right] da_y$$

$$[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})] = 0$$

$$\boxed{G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})}$$

(b) For Neumann boundary conditions, F can be chosen so that the simplest boundary condition (1.45) for G_N(**x**, **x'**) is:

$$\frac{\partial G_N}{\partial n'}(\mathbf{x},\mathbf{x}') = -\frac{4\pi}{S}$$ where S is the total surface area of the boundary.

The electrostatic Green function becomes:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}') G_N d^3\mathbf{x}' + \frac{1}{4\pi}\oint_S \left(G_N \frac{d\Phi}{d n'} \right) da' + \langle \Phi \rangle_S$$

The green function G_N in this case is not symmetric in general, shown by using the general form from above:

$$[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})] = -\frac{1}{4\pi}\oint_S \left[G(\mathbf{x},\mathbf{y}) \frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} \right] da_y$$

For Neumann boundary conditions, as stated above, $\frac{\partial G_N}{\partial n'} = -\frac{4\pi}{S}$ on the surface. The leads to:

$$[G_N(\mathbf{x},\mathbf{x}') - G_N(\mathbf{x}',\mathbf{x})] = \frac{1}{S}\oint_S G_N(\mathbf{x},\mathbf{y}) da_y - \frac{1}{S}\oint_S G_N(\mathbf{x}',\mathbf{y}) da_y$$

$$\boxed{G_N(\mathbf{x},\mathbf{x}') - \frac{1}{S}\oint_S G_N(\mathbf{x},\mathbf{y}) da_y = G_N(\mathbf{x}',\mathbf{x}) - \frac{1}{S}\oint_S G_N(\mathbf{x}',\mathbf{y}) da_y}$$

This is obviously not symmetric in general, but $G_N(\mathbf{x}, \mathbf{x}') - F(\mathbf{x})$ *is* symmetric in $\mathbf{x}$ and $\mathbf{x}'$, where

$$F(\mathbf{x}) = \frac{1}{S}\oint_S G_N(\mathbf{x}, \mathbf{y})\, da_y \quad .$$

(c) Start with the Neumann Green's function solution:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}') G_N\, d^3\mathbf{x}' + \frac{1}{4\pi}\oint_S \left(G_N \frac{d\Phi}{d\,n'} \right) da' + \langle\Phi\rangle_S$$

Now add to the Green function $F(\mathbf{x})$ and find its affect.

$$\Phi'(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')(G_N + F(\mathbf{x}))\, d^3\mathbf{x}' + \frac{1}{4\pi}\oint_S \left((G_N + F(\mathbf{x})) \frac{d\Phi}{d\,n'} \right) da' + \langle\Phi\rangle_S$$

$$\Phi'(\mathbf{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}') G_N\, d^3\mathbf{x}' + \frac{1}{4\pi}\oint_S \left(G_N \frac{d\Phi}{d\,n'} \right) da'$$
$$+ \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}') F(\mathbf{x})\, d^3\mathbf{x}' + \frac{1}{4\pi}\oint_S \left(F(\mathbf{x}) \frac{d\Phi}{d\,n'} \right) da'$$

$$\Phi'(\mathbf{x}) = \Phi(\mathbf{x}) + \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}') F(\mathbf{x})\, d^3\mathbf{x}' + \frac{1}{4\pi}\oint_S \left(F(\mathbf{x}) \frac{d\Phi}{d\,n'} \right) da'$$

$$\Phi'(\mathbf{x}) = \Phi(\mathbf{x}) + \frac{1}{4\pi} F(\mathbf{x}) \left[\frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}')\, d^3\mathbf{x}' + \oint_S \left(\frac{d\Phi}{d\,n'} \right) da' \right]$$

Use Gauss's Law in integral form in terms of a charge distribution (where all of the integration variables are primed to keep the notation consistent): $\oint_S \mathbf{E}\cdot\mathbf{n}'\, da' = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}')\, d^3\mathbf{x}'$

$$\Phi'(\mathbf{x}) = \Phi(\mathbf{x}) + \frac{1}{4\pi} F(\mathbf{x}) \left[\oint_S \mathbf{E}\cdot\mathbf{n}'\, da' + \oint_S \left(\frac{d\Phi}{d\,n'} \right) da' \right]$$

Use the definition of the scalar potential, $\mathbf{E} = -\nabla'\Phi$, and recognize that $\nabla'\Phi\cdot\mathbf{n}' = \frac{d\Phi}{dn'}$

$$\Phi'(\mathbf{x}) = \Phi(\mathbf{x}) + \frac{1}{4\pi} F(\mathbf{x}) \left[-\oint_S \left(\frac{d\Phi}{dn'} \right) da' + \oint_S \left(\frac{d\Phi}{d\,n'} \right) da' \right]$$

The last two terms now cancel so that

$$\Phi'(\mathbf{x}) = \Phi(\mathbf{x})$$

The addition of $F(\mathbf{x})$ to the Green function does not affect the potential $\Phi(\mathbf{x})$.