Jackson 1.13 Homework Problem Solution

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PROBLEM:

Two infinite grounded parallel conducting planes are separated by a distance *d*. A point charge *q* is placed between the planes. Use the reciprocation theorem of Green to prove that the total induced charge on one of the planes is equal to (-*q*) times the fractional perpendicular distance of the point charge from the other plane.

SOLUTION:

First of all, for definiteness, let us place the bottom plane at $z = -d/2$, the top plane at $z = d/2$, and the point charge at ($x = 0$, $y = 0$, $z = d/2 - a$) so that *a* is the distance between the top plate and the point charge.

Green's reciprocation theorem states that if Φ is the potential due to a volume-charge density ρ within a volume *V* and a surface-charge density σ on the conducting surface *S* bounding the volume *V*, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho \Phi' d^3x + \int_S \sigma \Phi' da = \int_V \rho' \Phi d^3x + \int_S \sigma' \Phi da$$

This theorem basically relates two separate problems that have different charge distributions but the same bounding surface. To take advantage of this theorem, we take the real problem as one of the problems, and then we can choose whatever problem we want as the second problem as long as it has the same bounding surface of two conducting planes.

For the second problem, let us choose that there is no charge density inside the volume, $\rho' = 0$, and that there is a uniform surface charge $+\sigma_0$ on the lower plane and a uniform surface charge $-\sigma_0$ on the upper plane. Because of symmetry and physical considerations, we immediately know that the electric field between these two planes is:

$$\mathbf{E}' = \frac{\sigma_0}{\epsilon_0}\hat{\mathbf{z}}$$

$$-\nabla \Phi' = \frac{\sigma_0}{\epsilon_0}\hat{\mathbf{z}}$$

Matching up vector components to find:

$$-\frac{d\Phi'}{dz} = \frac{\sigma_0}{\epsilon_0}$$

$$\Phi' = -\frac{\sigma_0}{\epsilon_0} z$$

Plugging in these charge densities and this potential into the theorem reduces it to:

$$-\frac{\sigma_0}{\epsilon_0}\int_V \rho z\, d^3x + \frac{\sigma_0}{\epsilon_0}\frac{d}{2}\int_{\text{bot}} \sigma\, da - \frac{\sigma_0}{\epsilon_0}\frac{d}{2}\int_{\text{top}} \sigma\, da = \sigma_0\left[\int_{\text{bot}} \Phi\, da - \int_{\text{top}} \Phi\, da\right]$$

The real problem has boundaries made out of grounded conductors, so that $\Phi = 0$ on the surfaces, making the terms on the right in the above equation go away.

$$-\int_V \rho z\, d^3x + \frac{d}{2}\int_{\text{bot}} \sigma\, da - \frac{d}{2}\int_{\text{top}} \sigma\, da = 0$$

We recognize the second and third integral as simply the total charge on the bottom and top planes respectively:

$$-\int_V \rho z\, d^3x + \frac{d}{2} q_{\text{bot}} - \frac{d}{2} q_{\text{top}} = 0$$

Finally, the charge distribution is simply a point charge at some point:

$$\rho = q\,\delta(x)\delta(y)\delta(z-(d/2-a))$$

$$-(d/2-a)q + \frac{d}{2} q_{\text{bot}} - \frac{d}{2} q_{\text{top}} = 0$$

$$(2\frac{a}{d}-1)q + q_{\text{bot}} - q_{\text{top}} = 0$$

Now, if were were to enclose this whole system in a Guassian surface, there would be no electric field on the Gaussian surface, because the planes are grounded. Therefore, the total charge in this system is zero:

$$q + q_{\text{top}} + q_{\text{bot}} = 0$$

$$q_{\text{top}} = -q - q_{\text{bot}}$$

Plugging this in:

$$\boxed{q_{\text{bot}} = -\frac{a}{d} q}$$

Therefore, the total charge on the bottom plate is (-*q*) times the fractional distance of the point charge from the top plate.