Jackson 1.12 Homework Problem Solution

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PROBLEM:

Prove *Green's reciprocation theorem*: If Φ is the potential due to a volume-charge density ρ within a volume *V* and a surface-charge density σ on the conducting surface *S* bounding the volume *V*, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi' d^3x + \int_S \sigma\Phi' da = \int_V \rho'\Phi d^3x + \int_S \sigma'\Phi\, da$$

SOLUTION:

Start with the Green's theorem:

$$\int_V \left(\Phi\nabla^2\Phi' - \Phi'\nabla^2\Phi\right) d^3x = \oint_S \left(\Phi\frac{d\Phi'}{dn} - \Phi'\frac{d\Phi}{dn}\right) da$$

Use the Poisson equation: $\nabla^2\Phi = -\frac{1}{\epsilon_0}\rho(\mathbf{x})$ and $\nabla^2\Phi' = -\frac{1}{\epsilon_0}\rho'(\mathbf{x})$:

$$-\frac{1}{\epsilon_0}\int_V \left(\rho'\Phi - \rho\Phi'\right) d^3x = \oint_S \left(\Phi\frac{d\Phi'}{dn} - \Phi'\frac{d\Phi}{dn}\right) da$$

$$\int_V \rho\Phi' d^3x - \int_V \rho'\Phi\, d^3x = \oint_S \left(\Phi\epsilon_0\frac{d\Phi'}{dn} - \Phi'\epsilon_0\frac{d\Phi}{dn}\right) da$$

The normal direction *n* in Green's theorem points *away* from the volume of interest, which in this case would be into the conducting surface. From a previous problem, we know that the potential on a conductor and surface charge density are related according to:

$$\left[\sigma = -\epsilon_0\frac{\partial\Phi}{\partial n}\right]_{\text{on S}} \quad \text{and} \quad \left[\sigma' = -\epsilon_0\frac{\partial\Phi'}{\partial n}\right]_{\text{on S}}$$

We have to realize that the normals in these equations are pointing out of the conductor, whereas the normals in Green's theorem are pointing into the conductor. We reverse the sign to account for this:

$$\left[\sigma = \epsilon_0\frac{\partial\Phi}{\partial n_{\text{in}}}\right]_{\text{on S}} \quad \text{and} \quad \left[\sigma' = \epsilon_0\frac{\partial\Phi'}{\partial n_{\text{in}}}\right]_{\text{on S}}$$

Now substitute these into Green's theorem:

$$\int_V \rho \Phi' d^3 x - \int_V \rho' \Phi d^3 x = \oint_S \left(\Phi \sigma' - \Phi' \sigma \right) da$$

Shuffle around to find:

$$\boxed{\int_V \rho \Phi' d^3 x + \int_S \sigma \Phi' da = \int_V \rho' \Phi d^3 x + \int_S \sigma' \Phi da}$$