



C. S. BAIRD

Jackson 1.11 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell



PROBLEM:

Use Gauss's theorem to prove that at the surface of a curved charged conductor, the normal derivative of the electric field is given by

$$\frac{1}{E} \frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where R_1 and R_2 are the principal radii of curvature of the surface.

SOLUTION:

We will find the normal derivative of the electric field by taking the limit of the finite difference:

$$\frac{\partial E}{\partial n} = \lim_{\Delta r \rightarrow 0} \frac{E(\mathbf{x} + \Delta r \hat{\mathbf{n}}) - E(\mathbf{x})}{\Delta r}$$

where Δr is a small length increment in the normal direction. This is the definition of a derivative according to the fundamental theorem of calculus.

We set a square Gaussian pillbox just above the surface of a point on the curved charged conductor (not straddling), and use Gauss's theorem to integrate over the pillbox. Make the upper and lower surfaces curved so that their curvature matches the conductor's surface. Set the location of the center of the lower surface at \mathbf{x} and the center of the upper surface at $\mathbf{x} + \Delta r \hat{\mathbf{n}}$. Set the sides of the pillbox normal to the conductor's surface so they do not contribute. The pillbox is above the surface and therefore contains no charge.

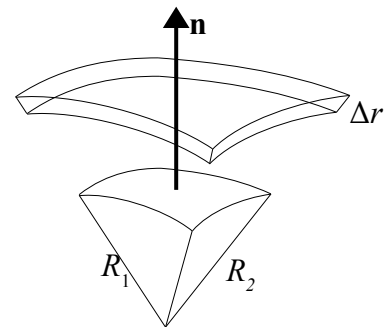
$$\oint_S \mathbf{E} \cdot \mathbf{n} da = 0$$

$$\int_{\text{top}} \mathbf{E} \cdot \mathbf{n} da + \int_{\text{bottom}} \mathbf{E} \cdot \mathbf{n} da = 0$$

Shrink the pillbox down in the usual way so that the electric field becomes constant across its surface and comes out of the integral. Be careful and remember that the normal to the bottom Gaussian surface is in the opposite direction as the conductor's normal, so we need a negative sign to account for this.

$$E_{\text{top}} \int_{\text{top}} da - E_{\text{bottom}} \int_{\text{bottom}} da = 0$$

$$E(\mathbf{x} + \Delta r \hat{\mathbf{n}})(R_1 + \Delta r)(R_2 + \Delta r) \int \int d\theta_1 d\theta_2 = E(\mathbf{x}) R_1 R_2 \int \int d\theta_1 d\theta_2$$



$$E(\mathbf{x} + \Delta r \hat{\mathbf{n}}) = \frac{E(\mathbf{x}) R_1 R_2}{(R_1 + \Delta r)(R_2 + \Delta r)}$$

Plug this into the normal derivative definition:

$$\frac{\partial E}{\partial n} = \lim_{\Delta r \rightarrow 0} \frac{\frac{R_1 R_2}{(R_1 + \Delta r)(R_2 + \Delta r)} - 1}{\Delta r} E$$

$$\frac{\partial E}{\partial n} = \lim_{\Delta r \rightarrow 0} \frac{-R_1 - R_2 - \Delta r}{(R_1 + \Delta r)(R_2 + \Delta r)} E$$

$$\boxed{\frac{1}{E} \frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

For very small curvatures of radius, such as the tip of a pointed conductor or the edge of a conducting cube, the derivative becomes very large. This means that the field is changing very quickly, so that the field lines are diverging away from the point or edge. For very large curvatures of radius, such as approaching a flat surface, this equation tells us the derivative approaches zero. This means that the electric field is approximately constant outside near-flat conducting surfaces.