Jackson 1.10 Homework Problem Solution

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PROBLEM:

Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point.

SOLUTION:

The potential is known on the surface, so this problem can be formulated using a Dirichlet Green's function equation:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}) G_D d^3\mathbf{x}' - \frac{1}{4\pi}\oint\left(\Phi\frac{d\,G_D}{d\,n'}\right)da'$$

where the Dirichlet Green's function must satisfy:

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}') \quad \text{where} \quad \nabla'^2 F(\mathbf{x},\mathbf{x}') = 0 \quad \text{and } G_D = 0 \text{ on the surface}$$

In this particular case, there is no free charge, $\rho(\mathbf{x}) = 0$, so that the equation simplifies to

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint\left(\Phi\frac{d\,G_D}{d\,n'}\right)da'$$

Because we are only measuring the potential at the center of the sphere which is centered on the origin, $\mathbf{x} = 0$ and therefore $\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{x'}$, leading to:

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{x'} + F(\mathbf{x},\mathbf{x}')$$

In order for the green function to disappear on the surface, $G_D(x' = R) = 0$, we must have $F = -1/R$. The Green function is now:

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{x'} - \frac{1}{R}$$

Insert this into the equation:

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint\left(\Phi\frac{d}{d\,x'}\left[\frac{1}{x'} - \frac{1}{R}\right]\right)_{x'=R} da'$$

$$\Phi(\mathbf{x})=\frac{1}{4\pi}\oint\left(\Phi\left[\frac{1}{x'^2}\right]\right)_{x'=R} da'$$

$$\Phi(\mathbf{x})=\frac{1}{4\pi}\oint\Phi\left[\frac{1}{R^2}\right]da'$$

$$\Phi(\mathbf{x})=\frac{\oint\Phi\, da'}{4\pi R^2}$$

The divisor is the surface area of the sphere so that:

$$\Phi(\mathbf{x})=\frac{\oint\Phi\, da'}{\oint da'}$$

The right side is by definition the average value of the function over the surface and thus equals the value at its center.