Jackson 11.3 Homework Problem Solution

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PROBLEM:

Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v=\frac{v_1+v_2}{1+(v_1 v_2/c^2)}$$

This is an alternative way to derive the parallel-velocity addition law.

SOLUTION:

The Lorentz transformations are:

$$x_0'=\gamma_1(x_0-\beta_1 x_1)$$

$$x_1'=\gamma_1(x_1-\beta_1 x_0) \quad \text{where} \quad \gamma_1=1/\sqrt{1-v_1^2/c^2} \quad , \quad \beta_1=v_1/c \quad \text{and } x_0 = ct$$

If we label another frame as the double-prime frame and define it as traveling at a speed v_2 relative to the prime frame, then the Lorentz transformation between these two frames is:

$$x_0''=\gamma_2(x_0'-\beta_2 x_1')$$

$$x_1''=\gamma_2(x_1'-\beta_2 x_0') \quad \text{where} \quad \gamma_2=1/\sqrt{1-v_2^2/c^2} \quad \text{, and} \quad \beta_2=v_2/c$$

If we now use the first Lorentz transformation as a definition of the prime variables and plug them into the second Lorentz transformation, we have:

$$x_0''=\gamma_2(\gamma_1(x_0-\beta_1 x_1)-\beta_2\gamma_1(x_1-\beta_1 x_0))$$

$$x_1''=\gamma_2(\gamma_1(x_1-\beta_1 x_0)-\beta_2\gamma_1(x_0-\beta_1 x_1))$$

Collect terms:

$$x_0''=\gamma_2\gamma_1((1+\beta_2\beta_1)x_0-(\beta_1+\beta_2)x_1)$$

$$x_1''=\gamma_2\gamma_1((1+\beta_2\beta_1)x_1-(\beta_1+\beta_2)x_0)$$

Now if we instead identified the double-primed frame as traveling at a speed v relative to the unprimed frame, then the Lorentz transformation relating the two would be:

$$x_0'' = \gamma(x_0 - \beta x_1)$$

$$x_1'' = \gamma(x_1 - \beta x_0) \quad \text{where} \quad \gamma = 1/\sqrt{1 - v^2/c^2} \quad \text{and} \quad \beta = v/c$$

Comparing this to the double transformation, we see that in order for them to be equivalent, the coefficients must match.

$$\gamma_2 \gamma_1 (1 + \beta_2 \beta_1) = \gamma \qquad \gamma_2 \gamma_1 (\beta_1 + \beta_2) = \beta\gamma \qquad \gamma_2 \gamma_1 (1 + \beta_2 \beta_1) = \gamma \qquad \gamma_2 \gamma_1 (\beta_1 + \beta_2) = \beta\gamma$$

It should be obvious that all of these equations are redundant. Let us take the first one, expand and solve for *v*.

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - v_2^2/c^2}} \frac{1}{\sqrt{1 - v_1^2/c^2}} \left(1 + \frac{v_2 v_1}{c^2}\right)$$

$$\frac{1}{1 - v^2/c^2} = \frac{1}{1 - v_2^2/c^2} \frac{1}{1 - v_1^2/c^2} \left(1 + \frac{v_2 v_1}{c^2}\right)^2$$

$$1 - v^2/c^2 = \frac{(1 - v_2^2/c^2)(1 - v_1^2/c^2)}{\left(1 + \frac{v_2 v_1}{c^2}\right)^2}$$

$$v = \sqrt{c^2 - \frac{(1 - v_2^2/c^2)(1 - v_1^2/c^2)c^2}{\left(1 + \frac{v_2 v_1}{c^2}\right)^2}}$$

$$v = \sqrt{c^2 - \frac{(c^2 - v_1^2 - v_2^2 + v_1^2 v_2^2/c^2)}{1 + v_1^2 v_2^2/c^4 + 2 v_1 v_2/c^2}}$$

$$\boxed{v = \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)}}$$