Jackson 11.1 Homework Problem Solution

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PROBLEM:

Two equivalent inertial frames K and K' are such that K' moves in the positive x direction with speed v as seen from K. The spatial coordinate axes in K' are parallel to those in K and the two origins are coincident at times $t = t' = 0$.

(a) Show that the isotropy and homogeneity of space-time and equivalence of different inertial frames (first postulate of relativity) require that the most general transformation between the space-time coordinates (x, y, z, t) and (x', y', z', t') is the linear transformation,

$$x' = f(v^2)x - v f(v^2)t; \quad t' = g(v^2)t - v h(v^2)x; \quad y' = y; \quad z' = z$$

and the inverse,

$$x = f(v^2)x' + v f(v^2)t'; \quad t = g(v^2)t' + v h(v^2)x'; \quad y = y'; \quad z = z'$$

where f, g, and h are functions of v^2, the structures of the x' and x equations are determined by the definition of the inertial frames in relative motion, and the signs of the inverse equation are a reflection of a reversal of roles of the two frames.

(b) Show that consistency of the initial transformation and its inverse requires

$$f = g \quad \text{and} \quad f^2 - v^2 f h = 1$$

(c) If a physical entity has speed u' parallel to the x' axis in K', show that its speed u parallel to the x axis in K is

$$u = \frac{u' + v}{1 + v u'(h/f)}$$

Using the second postulate 2' (universal limiting speed C), show that $h = f/C^2$ is required and that the Lorentz transformation of the coordinates results. The universal limiting speed C is to be determined from experiment.

SOLUTION:

(a) First of all, there is no motion in the y or z directions, so that these coordinates must be identical:

$$y' = y; \quad z' = z$$

Homogeneity of space-time means that points in space-time have the same density at all points as observed in one particular frame. Mathematically, this requires linear relationships between all space and time coordinates. (If, for instance, $x' = x^3$, than at locations marked by higher x, successive points in x' would be further and further spaced apart.) This gives:

$$x' = f x + f_2 t; \quad t' = g t + h x; \quad y' = y; \quad z' = z$$

At this point, f, f_2, g, and h are arbitrary functions that cannot depend on x or t. The only thing left that they could depend is the frame's velocity v and any universal constants that fall out.
We have inertial frames in relative motion, so that the shifting of the origins must be taken into account. In Galilean relativity, this was taken into account by adding $-vt$ to the spatial relation. We can get this in the same general form by taking a $-v$ out of the arbitrary f_2 and h:

$$x' = f x - v f_2 t; \quad t' = g t - v h x; \quad y' = y; \quad z' = z$$

Again, the homogenous nature of space-time means that as time marches on, the points cannot spread out. This forces upon us $f_2 = f$:

$$x' = f x - v f t; \quad t' = g t - v h x; \quad y' = y; \quad z' = z$$

If the functions f, g, h are functions of v, they should be functions of the magnitude of v and not the sign of v, so that the signs we have chosen above are preserved to keep the origins shifting in the right way. The most natural way to be the function of a variable's magnitude but not sign is to square it. We therefore assume that of the functions above depend at all on the velocity, they will depend on v^2:

$$x' = f(v^2) x - v f(v^2) t; \quad t' = g(v^2) t - v h(v^2) x; \quad y' = y; \quad z' = z$$

The isotropic nature of spacetime means that there is no special direction; the behavior of space-time is the same no matter which direction one frame is moving. This means that there are no special points and we could just as easily switch the primed variables with the unprimed variables and this expressions should still hold (as long as we are careful to to realize that the frame is now moving in the opposite direction, so we must switch $-v$ to $+v$).

$$x' = f(v^2) x - v f(v^2) t; \quad t' = g(v^2) t - v h(v^2) x; \quad y' = y; \quad z' = z$$

$$x = f(v^2) x' + v f(v^2) t'; \quad t = g(v^2) t' + v h(v^2) x'; \quad y = y'; \quad z = z'$$

(b) Show that consistency of the initial transformation and its inverse requires

$$f = g \quad \text{and} \quad f^2 - v^2 f h = 1$$

Consistency means that if we start in one inertial frame, transform to a different inertial frame, and then inverse transform back, we should end up in the exact same frame as the original frame, with all the space-time points at the same locations.

$$x = f x' + v f t'; \quad t = g t' + v h x'$$

$$x = f[f x - v f t] + v f[g t - v h x]; \quad t = g[g t - v h x] + v h[f x - v f t]$$

$$x = (f^2 - v^2 f h) x + (g - f) f v t; \quad t = (g^2 - v^2 h f) t + (f - g) h v x$$

In order for the first equation to reduce down to $x = x$ and the second equation to reduce down to $t = t$, we must have

$$f^2 - v^2 f h = 1 \quad \text{and} \quad f = g$$

which leads to:

$$h = \frac{(f^2 - 1)}{f v^2}$$

so that now the transformations reduce down to:

$$x' = f(v^2)(x - vt); \quad t' = f(v^2)\left(t - vx\frac{(f(v^2) - 1/f(v^2))}{f(v^2)v^2}\right); \quad y' = y; \quad z' = z$$

$$x = f(v^2)(x' + vt'); \quad t = f(v^2)\left(t' + vx'\frac{(f(v^2) - 1/f(v^2))}{f(v^2)v^2}\right); \quad y = y'; \quad z = z'$$

Now all we need to find is f. All we need is some kind of restriction or limitation or statement that will determine f. Notice that this is still quite general. In fact, this becomes Galilean relativity if we set $f = 1$.

(c) If a physical entity has speed u' parallel to the x' axis in K', show that its speed u parallel to the x axis in K is

$$u = \frac{u' + v}{1 + vu'(h/f)}$$

Using the second postulate 2' (universal limiting speed C), show that $h = f/C^2$ is required and that the Lorentz transformation of the coordinates results. The universal limiting speed C is to be determined from experiment.

The velocity of an object is just the change of position with respect to time:

$$u = \frac{dx}{dt}$$

The quantity dx is an incremental displacement in x and transforms in the exact same way as x. The same is true of dt.

$$u = \frac{f[dx' + v\,dt']}{f\left[dt' + v\,dx'\frac{(f - 1/f)}{f v^2}\right]}$$

$$u=\frac{u'+v}{1+vu'\frac{(f-1/f)}{f\,v^2}}$$

To check that this matches the form given in the question, we can go backwards and put *h* back in using what we had found it to be previously:

$$h=\frac{(f^2-1)}{f\,v^2}$$

$$u=\frac{u'+v}{1+vu'(h/f)}$$

Now that we know it matches, the more useful form is with *h* removed:

$$u=\frac{u'+v}{1+vu'\frac{(f-1/f)}{f\,v^2}}$$

The key step is to now postulate that there is a universal limiting speed *C* and no physical object in general can co faster than *C*. What that means is that if an object is going *C* in one frame, it must be going *C* in the other frame, as it can go no faster:

$$C=\frac{C+v}{1+vC\frac{(f-1/f)}{f\,v^2}}$$

Solve for *f*

$$f=\frac{1}{\sqrt{1-\left(\frac{v}{C}\right)^2}}$$

Plug back in:

$$x'=\frac{1}{\sqrt{1-\left(\frac{v}{C}\right)^2}}(x-vt); \qquad t'=\frac{1}{\sqrt{1-\left(\frac{v}{C}\right)^2}}\left(t-\frac{vx}{C^2}\right); \qquad y'=y; \qquad z'=z$$

$$x=\frac{1}{\sqrt{1-\left(\frac{v}{C}\right)^2}}(x'+vt'); \qquad t=\frac{1}{\sqrt{1-\left(\frac{v}{C}\right)^2}}\left(t'+\frac{vx'}{C^2}\right); \qquad y=y'; \qquad z=z'$$

Experimental measurements then reveal that the universal limiting speed limit *C* is indeed the speed of light in vacuum *c*. We thus see that it is not necessary to invoke the speed of light or that it is constant in all frames to get the Lorentz transformation, as long as we assume some universal limiting speed.