Jackson 11.14 Homework Problem Solution

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PROBLEM:

(a) Express the Lorentz scalars $F^{\alpha\beta}F_{\alpha\beta}$, $\mathscr{F}^{\alpha\beta}F_{\alpha\beta}$, and $\mathscr{F}^{\alpha\beta}\mathscr{F}_{\alpha\beta}$ in terms of **E** and **B**. Are there any other invariants quadratic in the field strengths **E** and **B**?

(b) Is it possible to have an electromagnetic field that appears as a purely electric field in one inertial frame and as a purely magnetic field in some other inertial frame? What are the criteria imposed on **E** and **B** such that there is an inertial frame in which there is no electric field?

(c) For macroscopic media, **E**, **B** form the field tensor $F^{\alpha\beta}$ and **D**, **H** the tensor $G^{\alpha\beta}$. What further invariants can be formed? What are their explicit expressions in terms of the 3-vector fields?

SOLUTION:

(a) The inner product of two four-tensors should give us a Lorentz scalar, i.e. a scalar that is the same in all frames.

$$F^{\alpha\beta}=\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

and

$$\mathscr{F}^{\alpha\beta}=\begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

The various inner products follow:

$$F^{\alpha\beta}F_{\alpha\beta}=\sum_{\alpha\beta\mu\nu}F^{\alpha\beta}g_{\alpha\mu}g_{\beta\nu}F^{\mu\nu}$$

Because $g_{\alpha\beta}=0$ for $\alpha\neq\beta$

$$F^{\alpha\beta}F_{\alpha\beta}=\sum_{\alpha\beta}F^{\alpha\beta}g_{\alpha\alpha}g_{\beta\beta}F^{\alpha\beta}$$

Separate the space components from the time components and use $g_{00}=1$, $g_{11}=g_{22}=g_{33}=-1$:

$$F^{\alpha\beta}F_{\alpha\beta}=F^{00}g_{00}g_{00}F^{00}+\sum_{\beta=1}^{3}F^{0\beta}g_{00}g_{\beta\beta}F^{0\beta}+\sum_{\alpha=1}^{3}F^{\alpha 0}g_{\alpha\alpha}g_{00}F^{\alpha 0}+\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}F^{\alpha\beta}g_{\alpha\alpha}g_{\beta\beta}F^{\alpha\beta}$$

$$F^{\alpha\beta}F_{\alpha\beta}=F^{00}F^{00}-\sum_{\beta=1}^{3}F^{0\beta}F^{0\beta}-\sum_{\alpha=1}^{3}F^{\alpha 0}F^{\alpha 0}+\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}F^{\alpha\beta}F^{\alpha\beta}$$

$$\boxed{F^{\alpha\beta}F_{\alpha\beta}=2\left(|\mathbf{B}|^2-|\mathbf{E}|^2\right)}$$

In a very similar way, we find:

$$\boxed{\mathcal{F}^{\alpha\beta}F_{\alpha\beta}=-4\,\mathbf{E}\cdot\mathbf{B}}$$

and

$$\boxed{\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}=\ \ -2\left(|\mathbf{B}|^2-|\mathbf{E}|^2\right)}$$

There are no other invariants quadratic in the field strengths **E** and **B** because there no other way to take the inner product of the field strength tensors.

(b) The fact that the electromagnetic field obeys Lorentz transformations means that the electric and magnetic field cannot have separate existences. What looks like a pure electric field in one frame will look like an electric and magnetic field in another frame. The questions then becomes: if we have a purely electric field in one inertial frame, can we find an inertial frame where it looks like a purely magnetic field? Let us look at the transformation:

$F'^{\alpha\beta}=\Lambda_{\gamma}^{\alpha}\Lambda_{\delta}^{\beta}F^{\gamma\delta}$ where the primed frame it moving at velocity **v** relative to the unprimed frame.

Written out in three-vector notation, the general transformation equations become:

$$\mathbf{E}'=\gamma\left(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}\right)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)$$

$$\mathbf{B}'=\gamma\left(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}\right)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)$$

In the unprimed frame, we have a purely electric field, $\mathbf{E}\neq 0$ and $\mathbf{B}=0$. These equations become:

$$\mathbf{E}'=\gamma\,\mathbf{E}-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)$$

$$\mathbf{B}'=-\gamma\,\boldsymbol{\beta}\times\mathbf{E}$$

The special frame where the purely electric field has become a purely magnetic field (it it exists) would require $\mathbf{E}'=0$, leading to:

$$\gamma \mathbf{E} = \frac{\gamma^2}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})$$

$$\hat{\mathbf{E}} E = \frac{\gamma}{\gamma+1} \beta \hat{\boldsymbol{\beta}}((\beta \hat{\boldsymbol{\beta}}) \cdot (\hat{\mathbf{E}} E))$$

$$\hat{\mathbf{E}} = \hat{\mathbf{v}} \quad \text{and} \quad 1 = \frac{\gamma}{\gamma+1} \frac{v^2}{c^2}$$

$$\boxed{v = c}$$

This tells us that the only frame with a purely magnetic field is the frame traveling at the speed of light relative to the initial frame. But nothing can go the speed of light that is initially at a different speed. Therefore this frame can never be reached and a purely magnetic field can never be attained from a purely electric one by switching frames.

Let us look at another way to approach the problem. We just showed that $|\mathbf{B}|^2 - |\mathbf{E}|^2 = C$ is the same in all frames. In the unprimed frame, **B** = 0, so that $C = -|\mathbf{E}|^2$. In the special frame (primed) we hope to find **E**' = 0 so that $C = |\mathbf{B}'|^2$. Any non-zero constant *C* cannot be both purely positive and purely negative at the same time (if *C* is zero, we have no fields in any frame, which negates the whole concept).

In order for there to be no electric field in a frame (**E'**= 0), the **E** field and **B** field in all other frames must obey:

$$0 = \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})$$

$$\boxed{\boldsymbol{\beta} \times \mathbf{B} = -\mathbf{E} + \frac{\gamma}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})}$$

(c) For fields in media, we have:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad \text{and} \quad G^{\alpha\beta} = \begin{bmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{bmatrix}$$

The possible inner products yielding Lorentz invariant scalars are:

$$\boxed{G^{\alpha\beta}G_{\alpha\beta}=2\left(|\mathbf{H}|^2-|\mathbf{D}|^2\right)}$$

$$\boxed{\mathcal{G}^{\alpha\beta}\mathcal{G}_{\alpha\beta}=-2\left(|\mathbf{H}|^2-|\mathbf{D}|^2\right)}$$

$$\boxed{\mathcal{G}^{\alpha\beta}G_{\alpha\beta}=-4\,\mathbf{D}\cdot\mathbf{H}}$$

$$\boxed{F^{\alpha\beta}G_{\alpha\beta}=2\left(\mathbf{B}\cdot\mathbf{H}-\mathbf{E}\cdot\mathbf{D}\right)}$$

$$\boxed{\mathcal{F}^{\alpha\beta}G_{\alpha\beta}=-2\left(\mathbf{D}\cdot\mathbf{B}+\mathbf{E}\cdot\mathbf{H}\right)}$$

$$\boxed{\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta}=-2\left(\mathbf{B}\cdot\mathbf{H}-\mathbf{E}\cdot\mathbf{D}\right)}$$

$$\boxed{F^{\alpha\beta}\mathcal{G}_{\alpha\beta}=-2\left(\mathbf{D}\cdot\mathbf{B}+\mathbf{E}\cdot\mathbf{H}\right)}$$

This exhausts all possible combinations of F, G, and their dual tensors. It should be obvious that many of the resulting Lorentz invariants are redundant. The independent Lorentz invariants from the lists in part (c) and part (a) are compiled to be (with trivial overall constants omitted):

$$\mathbf{E}\cdot\mathbf{B}\,,\ \mathbf{D}\cdot\mathbf{H}\,,\ |\mathbf{B}|^2-|\mathbf{E}|^2\,,\ |\mathbf{H}|^2-|\mathbf{D}|^2\,,\ \mathbf{B}\cdot\mathbf{H}-\mathbf{E}\cdot\mathbf{D}\,,\ \mathbf{D}\cdot\mathbf{B}+\mathbf{E}\cdot\mathbf{H}$$