Jackson 10.4 Homework Problem Solution

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PROBLEM:

An unpolarized wave of frequency $\omega = ck$ is scattered by a *slightly* lossy uniform isotropic dielectric sphere of radius R much smaller than a wavelength. The sphere is characterized by an ordinary real dielectric constant ε_r and a real conductivity σ. The parameters are such that the skin depth is very *large* compared to the radius R.

(a) Calculate the differential and total *scattering* cross sections.

(b) Show that the absorption cross section is:

$$\sigma_{\text{abs}} = 12\pi R^2 \frac{R Z_0 \sigma}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2}$$

(c) From part a write down the forward scattering amplitude and use the optical theorem to evaluate the total cross section. Compare your answer with the sum of the scattering and absorption cross sections from parts a and b. Comment.

SOLUTION:

(a) Because we are in the long-wavelength region, the problem reduces to the pseudo-static problem of a sphere in a uniform electric and magnetic field. The complex dielectric constant gives rise to an induced electric dipole moment as derived in class:

$$\mathbf{p} = 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} \right) R^3 \boldsymbol{\epsilon}_0 E_0$$

Because the sphere is only slightly lossy and the skin depth is so large, the effect of the conductivity giving rise to a magnetic dipole moment is negligible.

$$\epsilon/\epsilon_0 = \epsilon_r + i \frac{\sigma}{\omega \epsilon_0}$$

Using $\omega = c\,k$ and therefore $\omega = \frac{1}{\epsilon_0 Z_0} k$ we have:

$$\epsilon/\epsilon_0 = \epsilon_r + i \frac{Z_0 \sigma}{k}$$

Plugging this into the dipole induced in a purely dielectric sphere by a uniform electric field, we have:

$$\mathbf{p} = 4\pi\epsilon_0 \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k} \right) R^3 \boldsymbol{\epsilon}_0 E_0$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} \left| \boldsymbol{\epsilon}^* \cdot \mathbf{p} \right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{(\epsilon_r - 1)^2 + (Z_0 \sigma / k)^2}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} k^4 R^6 \left| \boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0 \right|^2$$

If unpolarized light is incident and unpolarized scattered light is measured, we sum over the averages:

$$\frac{d\sigma_{\text{unpol, unpol}}}{d\Omega} = \frac{1}{2} \left[\frac{d\sigma_{\text{HH}}}{d\Omega} + \frac{d\sigma_{\text{VH}}}{d\Omega} + \frac{d\sigma_{\text{HV}}}{d\Omega} + \frac{d\sigma_{\text{VV}}}{d\Omega} \right]$$

$$\boxed{\frac{d\sigma_{\text{unpol, unpol}}}{d\Omega} = \frac{(\epsilon_r - 1)^2 + (Z_0 \sigma / k)^2}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} k^4 R^6 \frac{1}{2} (1 + \cos^2\theta)}$$

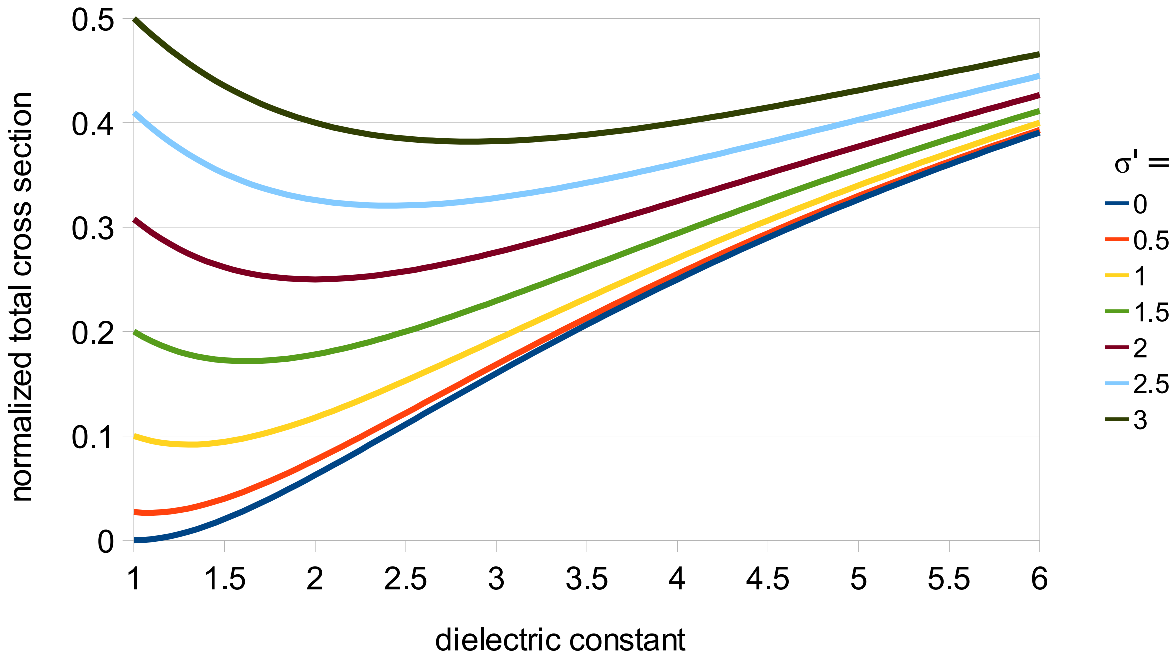
The total cross section is the integral over all angles in the usual way.

$$\boxed{\sigma_{\text{unpol, unpol}} = \frac{8\pi}{3} \frac{(\epsilon_r - 1)^2 + (Z_0 \sigma / k)^2}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} k^4 R^6}$$

We can put this in a format with dimensionless variables for the purpose of plotting:

$$\sigma_{\text{norm}} = \frac{(\epsilon_r - 1)^2 + \sigma'^2}{(\epsilon_r + 2)^2 + \sigma'^2} \quad \text{where} \quad \sigma_{\text{norm}} = \frac{\sigma_{\text{unpol, unpol}}}{\frac{8\pi}{3} k^4 R^6} \quad \text{and} \quad \sigma' = Z_0 \sigma / k$$

The dielectric constant ϵ_r will be our *x* axis, the normalized total cross section σ_{norm} will be our *y* axis, and the normalized conductivity σ' will be the different colored curves.



(b) The absorption cross section is defined as the total power absorbed over the incident power per unit area:

$$\sigma_{abs} = \frac{P_{abs}}{|E_0|^2 / 2 Z_0}$$

The total power absorbed is just the integral over the object's surface of the inward component of the energy flow vector:

$$P_{abs} = -\frac{1}{2\mu_0} \oint_S \Re(\mathbf{E} \times \mathbf{B}^*) \cdot \mathbf{n}' \, d a'$$

In our case, the surface is a sphere:

$$P_{abs} = -\frac{R^2}{2\mu_0} \int_0^{2\pi} \int_0^{\pi} \Re(\mathbf{E} \times \mathbf{B}^*) \cdot \hat{\mathbf{r}} \sin\theta \, d\theta \, d\phi$$

The absorption cross section therefore becomes:

$$\sigma_{abs} = -\frac{c R^2}{|E_0|^2} \int_0^{2\pi} \int_0^{\pi} \Re(\mathbf{E} \times \mathbf{B}^*) \cdot \hat{\mathbf{r}} \sin\theta \, d\theta \, d\phi$$

We have to be careful to realize these are total fields – incident plus scattered – because part of the incident field that flows in becomes the scattered field that flows out:

$$\sigma_{\text{abs}} = -\frac{c R^2}{|E_0|^2} \int_0^{2\pi} \int_0^{\pi} \Re\left((\mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{scat}}) \times (\mathbf{B}_{\text{inc}}{}^* + \mathbf{B}_{\text{scat}}{}^*) \right) \cdot \hat{\mathbf{r}} \sin\theta \, d\theta \, d\phi$$

The incident field is $\mathbf{E}_{\text{inc}} = \boldsymbol{\epsilon}_0 E_0 e^{i \mathbf{k}_0 \cdot \mathbf{x}}$.

The scattered field is:

$$\mathbf{E}_{\text{sc}} = -\frac{k^2}{4\pi\epsilon_0} \left[\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{p}) \right] \frac{e^{i \mathbf{k}\cdot\mathbf{x}}}{r}$$

$$\mathbf{E}_{\text{sc}} = -E_0 \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k} \right) k^2 R^3 \left[\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) \right] \frac{e^{i \mathbf{k}\cdot\mathbf{x}}}{r}$$

The magnetic fields obey: $\mathbf{B} = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E}$ and are therefore:

$$\mathbf{B}_{\text{inc}} = \frac{1}{c} E_0 e^{i \mathbf{k}_0 \cdot \mathbf{x}} \hat{\mathbf{k}}_0 \times \boldsymbol{\epsilon}_0$$

$$\mathbf{B}_{\text{scat}} = \frac{1}{c} E_0 \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k} \right) k^2 R^3 (\hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) \frac{e^{i \mathbf{k}\cdot\mathbf{x}}}{r}$$

Plugging these all in:

$$\sigma_{\text{abs}} = -R^2 \int_0^{2\pi} \int_0^{\pi} \Re\left(\left(\boldsymbol{\epsilon}_0 e^{i \mathbf{k}_0 \cdot \mathbf{x}} - \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k} \right) k^2 R^2 \left[\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) \right] e^{i \mathbf{k}\cdot\mathbf{x}} \right) \right.$$
$$\left. \times \left(e^{-i \mathbf{k}_0 \cdot \mathbf{x}} \hat{\mathbf{k}}_0 \times \boldsymbol{\epsilon}_0 + \left(\frac{(\epsilon_r - 1) - i Z_0 \sigma / k}{(\epsilon_r + 2) - i Z_0 \sigma / k} \right) k^2 R^2 (\hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) e^{-i \mathbf{k}\cdot\mathbf{x}} \right) \right) \cdot \hat{\mathbf{r}} \sin\theta \, d\theta \, d\phi$$

$$\sigma_{\text{abs}} = -R^2 \int_0^{2\pi} \int_0^{\pi} \Re\left(\left(\left(\hat{\mathbf{k}}_0 + \left(\frac{(\epsilon_r - 1) - i Z_0 \sigma / k}{(\epsilon_r + 2) - i Z_0 \sigma / k} \right) k^2 R^2 (\boldsymbol{\epsilon}_0 \times \hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) e^{i (\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}} \right) \right. \right.$$
$$- \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k} \right) k^2 R^2 e^{-i (\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}} \left[\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) \right] \times \hat{\mathbf{k}}_0 \times \boldsymbol{\epsilon}_0$$
$$\left. \left. - \left(\frac{(\epsilon_r - 1)^2 + (Z_0 \sigma / k)^2}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} \right) k^4 R^4 \left[\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) \right] \times (\hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) \right) \right) \cdot \hat{\mathbf{r}} \sin\theta \, d\theta \, d\phi$$

The first term integrates to zero. The last term can be dropped because the radius is very small compared to the skin depth:

$$\sigma_{\text{abs}} = -\frac{k^2 R^4}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} \int_0^{2\pi} \int_0^{\pi} \Re\left(\left(\left(\left((\epsilon_r - 1) - i Z_0 \sigma / k \right) \left((\epsilon_r + 2) + i Z_0 \sigma / k \right) \right) (\boldsymbol{\epsilon}_0 \times \hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) e^{i k R (\cos\theta - 1)} \right) \right.$$
$$\left. - \left(\left((\epsilon_r - 1) + i Z_0 \sigma / k \right) \left((\epsilon_r + 2) - i Z_0 \sigma / k \right) \right) e^{-i k R (\cos\theta - 1)} \left[\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times (\boldsymbol{\epsilon}_0)) \right] \times \hat{\mathbf{k}}_0 \times \boldsymbol{\epsilon}_0 \right) \cdot \hat{\mathbf{r}} \sin\theta \, d\theta \, d\phi$$

$$\sigma_{\text{abs}} = -\frac{k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}\int_0^{2\pi}\int_0^{\pi}\left(\left(\left(\left(\left(A\cos(kR(\cos\theta-1))+3Z_0\sigma/k\sin(kR(\cos\theta-1))\right)\right)\right)(\boldsymbol{\epsilon}_0\times\hat{\mathbf{k}}\times(\boldsymbol{\epsilon}_0))\right)\right)$$
$$-\left(\left(A\cos(kR(\cos\theta-1))+3Z_0\sigma/k\sin(kR(\cos\theta-1))\right)\right)\left[\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times(\boldsymbol{\epsilon}_0))\right]\times\hat{\mathbf{k}}_0\times\boldsymbol{\epsilon}_0)\cdot\hat{\mathbf{r}}\sin\theta\, d\theta\, d\phi$$

where $A=((\epsilon_r-1)(\epsilon_r+2)+(Z_0\sigma/k)^2)$

Now work out the vectors for the H and V incident polarization possibilities and average over them

$$\sigma_{\text{abs}} = -\frac{\pi k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}\int_0^{\pi}\left(\left(\left(\left(\left(A\cos(kR(\cos\theta-1))+3Z_0\sigma/k\sin(kR(\cos\theta-1))\right)\right)\right)(1+\cos^2\theta)\right)\right)$$
$$+2\left(\left(A\cos(kR(\cos\theta-1))+3Z_0\sigma/k\sin(kR(\cos\theta-1))\right)\right)\cos\theta)\sin\theta\, d\theta$$

Make the substitution $x = \cos\theta$, $dx = -\sin\theta\, d\theta$

$$\sigma_{\text{abs}} = -\frac{\pi k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}\int_{-1}^{1}\left(\left(\left(\left(\left(A\cos(kR(x-1))+3Z_0\sigma/k\sin(kR(x-1))\right)\right)\right)(1+x^2)\right)\right)$$
$$+2\left(\left(A\cos(kR(x-1))+3Z_0\sigma/k\sin(kR(x-1))\right)\right)x)\,dx$$

$$\sigma_{\text{abs}} = -\frac{\pi k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}[(A\int_{-2}^{0}x^2\cos(kRx)\,dx+4A\int_{-2}^{0}x\cos(kRx)\,dx+4A\int_{-2}^{0}\cos(kRx)\,dx$$
$$+3Z_0\sigma/k\int_{-2}^{0}x^2\sin(kRx)\,dx+12Z_0\sigma/k\int_{-2}^{0}x\sin(kRx)\,dx+12Z_0\sigma/k\int_{-2}^{0}\sin(kRx)\,dx]$$

$$\sigma_{\text{abs}} = -\frac{\pi k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}[(-2A\sin(2kR)/k^3R^3+4A/k^2R^2+Z_0\sigma/k[6(1-\cos(2kR))/k^3R^3-12/kR]]$$

Drop all higher order terms to find:

$$\boxed{\sigma_{\text{abs}} = 12\pi R^2\frac{RZ_0\sigma}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}}$$

The total cross section is:

$$\boxed{\sigma_{\text{tot}} = \frac{4\pi R^3}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}[3Z_0\sigma+\frac{2}{3}(\epsilon_r-1)^2k^4R^2+\frac{2}{3}Z_0^2\sigma^2k^2R^2]}$$

c) The optical theorem states:

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\Im\left[\frac{\boldsymbol{\epsilon}_0\cdot\mathbf{F}(\mathbf{k}=\mathbf{k}_0)}{E_0}\right]$$

$$\sigma_{\text{tot}} = \frac{-4\pi}{k} \Im\left[\boldsymbol{\epsilon}_0 \cdot \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k} \right) k^2 R^3 \left[\hat{\mathbf{k}}_0 \times (\hat{\mathbf{k}}_0 \times (\boldsymbol{\epsilon}_0)) \right] \right]$$

$$\boxed{\sigma_{\text{tot}} = 12\pi R^2 \frac{R Z_0 \sigma}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2}}$$

Interestingly, this only matches the result from part b if the absorption dominates so much over the scattering that the scattering becomes negligible. This makes sense because the higher orders can be dropped for a small sphere in the long-wavelength.