Jackson 10.3 Homework Problem Solution

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PROBLEM:
A solid uniform sphere of radius R and conductivity σ acts as a scatterer of a plane-wave beam of unpolarized radiation of frequency ω, with $\omega R/c << 1$. The conductivity is large enough that the skin depth δ is small compared to R.

(a) Justify and use a magnetostatic scalar potential to determine the magnetic field around the sphere, assuming the conductivity is infinite. (Remember that $\omega \neq 0$.)

(b) Use the technique of Section 8.1 to determine the absorption cross section of the sphere. Show that it varies as $(\omega)^{1/2}$ provided σ is independent of frequency.

SOLUTION:
(a) The statement $\omega R/c << 1$ is equivalent to $R << \lambda$. This is the long-wavelength region. In this region, we can safely assume that the fields are constant across the object, so that the problem reduces to a electrostatics/magnetostatics problem. In this case, we want to find the absorption, which is caused by induced currents meeting resistance. So we first need to find the currents. So we choose to approach this problem as a magnetostatics problem, because static magnetic fields are linked to currents.

A perfectly conducting sphere is placed in an originally uniform magnetic field.

The region of interest, outside the sphere, is free space and contains no charges or currents, so that we immediately know:

$$\nabla\cdot\mathbf{B}=0 \quad , \quad \nabla\times\mathbf{B}=0 \quad \text{(Because } \mathbf{B}=\mu_0\mathbf{H} \text{ and } \mathbf{J}=0\text{)}$$

The curl being zero (the second equation) lets us define the magnetic field in terms of the gradient of a scalar potential:

$$\mathbf{B}=-\nabla\Psi_M$$

Plugging this into the first equation leads to:

$$\nabla^2\Psi_M=0$$

This is just the familiar Laplace equation. If we place the sphere at the origin, and set the original field pointing along the z-axis, the problem is azimuthally symmetric. The solution to the Laplace equation in spherical coordinates with azimuthal symmetry is:

$$\Psi_M=\sum_{l=0}^{\infty}[A_l r^l+B_l r^{-l-1}]P_l(\cos\theta)$$

Apply the boundary condition at infinity:

$$\mathbf{B}(r=\infty)=B_0\hat{\mathbf{z}}$$

$$\Psi_M(r=\infty)=-B_0 z$$

$$\Psi_M(r=\infty)=-B_0 r\cos\theta$$

$$-B_0 r\cos\theta=\sum_{l=0}^{\infty} A_l(\infty)^l P_l(\cos\theta)$$

$$-B_0 r\cos\theta=A_0+A_1 r\cos\theta+\sum_{l=2}^{\infty} A_l(\infty)^l P_l(\cos\theta)$$

$$A_0=0,\ A_1=-B_0,\ A_l=0 \text{ for } l>1$$

Our solution now becomes:

$$\Psi_M=-B_0 r\cos\theta+\sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta)$$

For the next boundary condition, we remember that on the surface of a perfect conductor, the normal component of the magnetic field goes to zero:

$$\mathbf{B}(r=R)\cdot\hat{\mathbf{r}}=0$$

$$\left[\frac{\partial\Psi_M}{\partial r}=0\right]_{r=R}$$

$$\left[\left(-B_0\cos\theta+\sum_{l=0}^{\infty} B_l(-l-1)r^{-l-2} P_l(\cos\theta)\right)=0\right]_{r=R}$$

$$-B_0\cos\theta+\sum_{l=0}^{\infty} B_l(-l-1)R^{-l-2} P_l(\cos\theta)=0$$

$$-B_0\cos\theta+B_0(-1)R^{-2}+B_1(-2)R^{-3}\cos\theta+\sum_{l=2}^{\infty} B_l(-l-1)R^{-l-2} P_l(\cos\theta)=0$$

$$B_0=0,\ B_1=-R^3 B_0/2,\ B_l=0 \text{ for } l>1$$

Our solution now becomes:

$$\boxed{\Psi_M=-B_0 r\cos\theta-\frac{1}{2}R^3 B_0 r^{-2}\cos\theta}$$

$$\mathbf{B}=-\nabla\Psi_M$$

$$\boxed{\mathbf{B}=\hat{\mathbf{r}}\,B_0\cos\theta\left(1-\left(\frac{R}{r}\right)^3\right)-\hat{\boldsymbol{\theta}}\,B_0\sin\theta\left(1+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)}$$

(b) Section 8.1 tells us that the power loss due to absorption is dependent on the induced effective surface currents $\mathbf{K}_{\text{eff}}$ which in turn is dependent on the tangential surface magnetic field, so that:

$$\frac{d\,P_{\text{loss}}}{d\,a}=\frac{\mu_c\,\omega\,\delta}{4}\left|\mathbf{H}_{\text{par}}\right|^2$$

$$\frac{d\,P_{\text{loss}}}{d\,a}=\frac{\mu_c\,\omega\,\delta}{4\mu_0^2}\left|\mathbf{B}_{\text{par}}\right|^2$$

$$\frac{d\,P_{\text{loss}}}{d\,a}=\frac{\mu_c\,\omega\,\delta}{4\mu_0^2}\left|\mathbf{B}\cdot\hat{\boldsymbol{\theta}}\right|^2$$

$$\frac{d\,P_{\text{loss}}}{d\,a}=\frac{9\,\mu_c\,\omega\,\delta}{16\mu_0^2}\,B_0^2\sin^2\theta$$

To find the total absorption cross section of the sphere, we integrate this over the sphere:

$$P_{\text{loss}}=2\pi R^2\frac{9\,\mu_c\,\omega\,\delta}{16\mu_0^2}\,B_0^2\int_0^{\pi}\sin^2\theta\sin\theta\,d\,\theta$$

$$P_{\text{loss}}=\frac{3\pi R^2\mu_c\,\omega\,\delta}{2\mu_0^2}\,B_0^2$$

$$\sigma_{\text{abs}}=\frac{P_{\text{loss}}}{I_0}$$

$$\sigma_{\text{abs}}=\frac{P_{\text{loss}}}{\frac{1}{2}\sqrt{\frac{1}{\epsilon_0\mu_0^3}}\,B_0^2}$$

$$\boxed{\sigma_{\text{abs}}=3\pi R^2\sqrt{\frac{2\,\epsilon_0\,\omega\,\mu_c}{\sigma\mu_0}}}$$

If the sphere is nonmagnetic, this reduces down to:

$$\boxed{\sigma_{\text{abs}}=3\pi R^2\sqrt{\frac{2\,\epsilon_0\,\omega}{\sigma}}}$$