Jackson 10.1 Homework Problem Solution

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PROBLEM:

(a) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius *a*, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_0, \mathbf{n}_0, \mathbf{n}) = k^4 a^6 \left[\frac{5}{4} - |\boldsymbol{\epsilon}_0 \cdot \mathbf{n}|^2 - \frac{1}{4}|\mathbf{n}\cdot(\mathbf{n}_0 \times \boldsymbol{\epsilon}_0)|^2 - \mathbf{n}_0 \cdot \mathbf{n}\right]$$

where $\mathbf{n}_0$ and $\mathbf{n}$ are the directions of the incident and scattered radiations, respectively, while $\boldsymbol{\varepsilon}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\boldsymbol{\varepsilon}_0^* \cdot \boldsymbol{\varepsilon}_0 = 1$, $\mathbf{n}_0 \cdot \boldsymbol{\varepsilon}_0 = 0$).

(b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_0, \mathbf{n}_0, \mathbf{n}) = k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta \cos 2\phi\right]$$

where $\mathbf{n}\cdot\mathbf{n}_0 = \cos\theta$ and the azimuthal angle φ is measured from the direction of the linear polarization.

(c) What is the ratio of scattered intensities at $\theta = \pi/2$, $\varphi = 0$ and $\theta = \pi/2$, $\varphi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

SOLUTION:

The differential scattering cross section with the usual approximations was found to be:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\epsilon}^* \cdot \mathbf{p} + \frac{1}{c}(\mathbf{n}\times\boldsymbol{\epsilon}^*)\cdot\mathbf{m}\right|^2$$

The dipole moments induced by a plane wave on a perfectly conducting sphere are:

$$\mathbf{p} = 4\pi\epsilon_0 a^3 E_0 \boldsymbol{\epsilon}_0 \qquad \mathbf{m} = -2\pi a^3 \frac{1}{\mu_0 c} E_0 \mathbf{n}_0 \times \boldsymbol{\epsilon}_0$$

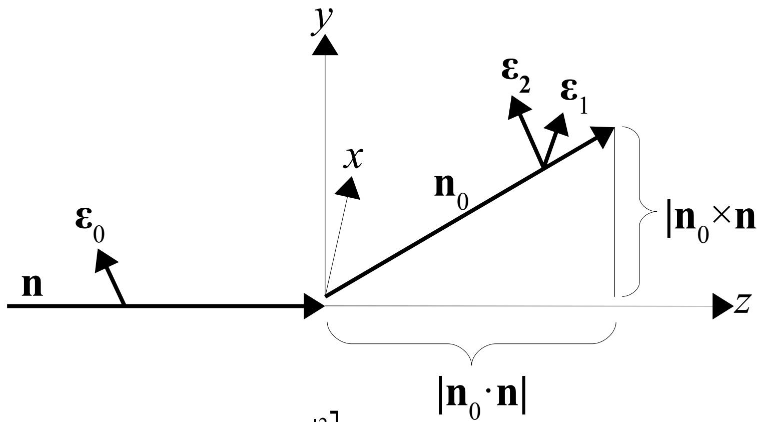
Plugging these in above gives:

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0 - \frac{1}{2}(\mathbf{n}\times\boldsymbol{\epsilon}^*)\cdot(\mathbf{n}_0 \times \boldsymbol{\epsilon}_0)\right|^2$$

Let us now sum over outgoing polarizations. Because the end goal is just to find the sum of outgoing polarizations, we can choose any orthogonal basis we want for the scattered wave. Let us therefor choose linear polarization where one basis vector is in the scattering plane and one is normal to it. By choosing linear polarization for the scattered wave, they are real valued which will make the math easier. The only complex-valued variable is the incident polarization $\boldsymbol{\varepsilon}_0^*$. Labeling the two possibilities for $\boldsymbol{\varepsilon}$ as $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$. Then $\boldsymbol{\epsilon}_2=\mathbf{n}\times\boldsymbol{\epsilon}_1$ and a little geometry and reflection lets us derive that:

$$\boldsymbol{\epsilon}_1=\frac{\mathbf{n}\times\mathbf{n}_0}{|\mathbf{n}\times\mathbf{n}_0|} \quad \text{and} \quad \boldsymbol{\epsilon}_2=\frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0)}{|\mathbf{n}\times\mathbf{n}_0|}$$

$$|\mathbf{n}\cdot\mathbf{n}_0|^2+|\mathbf{n}\times\mathbf{n}_0|^2=1$$



Sum over both polarizations:

$$\frac{d\sigma}{d\Omega}=k^4a^6\left[\left|\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_0-\frac{1}{2}(\mathbf{n}\times\boldsymbol{\epsilon}_1)\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)\right|^2+\left|\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_0-\frac{1}{2}(\mathbf{n}\times\boldsymbol{\epsilon}_2)\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)\right|^2\right]$$

Use the orthogonality of the scattered direction and polarization vectors to simplify

$$\frac{d\sigma}{d\Omega}=k^4a^6\left[\left|\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_0-\frac{1}{2}\boldsymbol{\epsilon}_2\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)\right|^2+\left|\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_0+\frac{1}{2}\boldsymbol{\epsilon}_1\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)\right|^2\right]$$

Plug in the definitions above:

$$\frac{d\sigma}{d\Omega}=k^4a^6\left[\left|\frac{\mathbf{n}\times\mathbf{n}_0}{|\mathbf{n}\times\mathbf{n}_0|}\cdot\boldsymbol{\epsilon}_0-\frac{1}{2}\frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0)}{|\mathbf{n}\times\mathbf{n}_0|}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)\right|^2+\left|\frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0)}{|\mathbf{n}\times\mathbf{n}_0|}\cdot\boldsymbol{\epsilon}_0+\frac{1}{2}\frac{\mathbf{n}\times\mathbf{n}_0}{|\mathbf{n}\times\mathbf{n}_0|}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)\right|^2\right]$$

$$\frac{d\sigma}{d\Omega}=k^4a^6\frac{1}{|\mathbf{n}\times\mathbf{n}_0|^2}\left[\left|(\mathbf{n}\times\mathbf{n}_0)\cdot\boldsymbol{\epsilon}_0-\frac{1}{2}(\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0))\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)\right|^2+\left|(\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0))\cdot\boldsymbol{\epsilon}_0+\frac{1}{2}(\mathbf{n}\times\mathbf{n}_0)\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)\right|^2\right]$$

Expand the squares

$$\begin{aligned}\frac{d\sigma}{d\Omega}=k^4a^6\frac{1}{|\mathbf{n}\times\mathbf{n}_0|^2}[&|(\mathbf{n}\times\mathbf{n}_0)\cdot\boldsymbol{\epsilon}_0|^2+\frac{1}{4}|(\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0))\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2-\frac{1}{2}((\mathbf{n}\times\mathbf{n}_0)\cdot\boldsymbol{\epsilon}_0^*)((\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0))\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0))\\ &\frac{-1}{2}((\mathbf{n}\times\mathbf{n}_0)\cdot\boldsymbol{\epsilon}_0)((\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0))\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0^*))+|(\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0))\cdot\boldsymbol{\epsilon}_0|^2+\frac{1}{4}|(\mathbf{n}\times\mathbf{n}_0)\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2+\\ &\frac{1}{2}((\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0))\cdot\boldsymbol{\epsilon}_0^*)((\mathbf{n}\times\mathbf{n}_0)\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0))+\frac{1}{2}((\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_0))\cdot\boldsymbol{\epsilon}_0)((\mathbf{n}\times\mathbf{n}_0)\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0^*))]\end{aligned}$$

Use several vector identities and the orthogonality of several of the vectors to simplify:

$$\begin{aligned}\frac{d\sigma}{d\Omega}=k^4a^6\frac{1}{|\mathbf{n}\times\mathbf{n}_0|^2}[&|\mathbf{n}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2+\frac{1}{4}|(\mathbf{n}_0\cdot\mathbf{n})|^2|\mathbf{n}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2-|(\mathbf{n}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0))|^2(\mathbf{n}\cdot\mathbf{n}_0)\\ &+|\mathbf{n}_0\cdot\mathbf{n}|^2|\boldsymbol{\epsilon}_0\cdot\mathbf{n}|^2+\frac{1}{4}|\boldsymbol{\epsilon}_0\cdot\mathbf{n}|^2-\mathbf{n}_0\cdot\mathbf{n}|\boldsymbol{\epsilon}_0\cdot\mathbf{n}|^2]\end{aligned}$$

Note that using the Pythagorean theorem we can show that $|\mathbf{n}\cdot\mathbf{n}_0|^2 = 1 - |\mathbf{n}\times\mathbf{n}_0|^2$. Plug this in

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\frac{\left(\frac{5}{4} - \mathbf{n}_0\cdot\mathbf{n}\right)\left(|\mathbf{n}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2 + |\boldsymbol{\epsilon}_0\cdot\mathbf{n}|^2\right)}{|\mathbf{n}\times\mathbf{n}_0|^2} - \frac{1}{4}|\mathbf{n}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2 - |\boldsymbol{\epsilon}_0\cdot\mathbf{n}|^2 \right]$$

Now note that a right-handed orthonormal set is formed by the vectors

$$\mathbf{n}_0\,,\ \boldsymbol{\epsilon}_0,\ \mathbf{n}_0\times\boldsymbol{\epsilon}_0$$

Any vector **b** can be expanded in this basis:

$$\mathbf{b} = (\mathbf{b}\cdot\mathbf{n}_0)\,\mathbf{n}_0 + (\mathbf{b}\cdot\boldsymbol{\epsilon}_0)\boldsymbol{\epsilon}_0 + (\mathbf{b}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0))(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)$$

and the magnitude squared of the vector **b** is just the sum of the square of its components:

$$b = |\mathbf{b}\cdot\mathbf{n}_0|^2 + |\mathbf{b}\cdot\boldsymbol{\epsilon}_0|^2 + |\mathbf{b}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2$$

Let us choose the unit-magnitude vector **n** as our **b** in this relation:

$$1 = |\mathbf{n}\cdot\mathbf{n}_0|^2 + |\mathbf{n}\cdot\boldsymbol{\epsilon}_0|^2 + |\mathbf{n}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2$$

Combine this with $|\mathbf{n}\cdot\mathbf{n}_0|^2 = 1 - |\mathbf{n}\times\mathbf{n}_0|^2$ to get

$$|\mathbf{n}\times\mathbf{n}_0|^2 = |\mathbf{n}\cdot\boldsymbol{\epsilon}_0|^2 + |\mathbf{n}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2$$

This cancels out the denominator in the first term so that finally:

$$\boxed{\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_{0,}\,\mathbf{n}_{0,}\,\mathbf{n}) = k^4 a^6 \left[\frac{5}{4} - |\boldsymbol{\epsilon}_0\cdot\mathbf{n}|^2 - \frac{1}{4}|\mathbf{n}\cdot(\mathbf{n}_0\times\boldsymbol{\epsilon}_0)|^2 - \mathbf{n}_0\cdot\mathbf{n}\right]}$$

(b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_{0,}\,\mathbf{n}_{0,}\,\mathbf{n}) = k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]$$

where $\mathbf{n}\cdot\mathbf{n}_0 = \cos\theta$ and the azimuthal angle φ is measured from the direction of the linear polarization.

The linear incident polarization vector can be broken down into components parallel and perpendicular to the scattering plane:

$$\boldsymbol{\epsilon}_0 = \sin\phi\,\boldsymbol{\epsilon}_{0,H} + \cos\phi\,\boldsymbol{\epsilon}_{0,V}$$

Plugging this in

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_{0,}\mathbf{n}_{0,}\mathbf{n})=k^4a^6\left[\frac{5}{4}-|(\sin\phi\,\boldsymbol{\epsilon}_{0,H}+\cos\phi\,\boldsymbol{\epsilon}_{0,V})\cdot\mathbf{n}|^2-\frac{1}{4}|\mathbf{n}\cdot(\mathbf{n}_0\times(\sin\phi\,\boldsymbol{\epsilon}_{0,H}+\cos\phi\,\boldsymbol{\epsilon}_{0,V}))|^2-\mathbf{n}_0\cdot\mathbf{n}\right]$$

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_{0,}\mathbf{n}_{0,}\mathbf{n})=k^4a^6\left[\frac{5}{4}-\cos^2\phi\sin^2\theta-\frac{1}{4}\sin^2\phi\sin^2\theta-\cos\theta\right]$$

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_{0,}\mathbf{n}_{0,}\mathbf{n})=k^4a^6\left[\frac{5}{4}-(\frac{1}{2}+\frac{1}{2}\cos2\phi)\sin^2\theta-\frac{1}{4}(\frac{1}{2}-\frac{1}{2}\cos2\phi)\sin^2\theta-\cos\theta\right]$$

$$\boxed{\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_{0,}\mathbf{n}_{0,}\mathbf{n})=k^4a^6\left[\frac{5}{8}(1+\cos^2\theta)-\cos\theta-\frac{3}{8}\sin^2\theta\cos2\phi\right]}$$

(c) What is the ratio of scattered intensities at $\theta = \pi/2$, $\varphi = 0$ and $\theta = \pi/2$, $\varphi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

$$\frac{\frac{d\sigma_V}{d\Omega}}{\frac{d\sigma_H}{d\Omega}}=\frac{k^4a^6\left[\frac{5}{8}-\frac{3}{8}\right]}{k^4a^6\left[\frac{5}{8}+\frac{3}{8}\right]}$$

$$\frac{\frac{d\sigma_V}{d\Omega}}{\frac{d\sigma_H}{d\Omega}}=\frac{1}{4}$$

When $\varphi = 0$, the polarization of the incident wave is in the scattering plane. At $\theta = \pi/2$ we are thus measuring along the axis of the induced *electric* dipole moment. Dipoles do not radiate along their axis, so in this case, the scattered radiation is completely due to the magnetic dipole. Conversely, when $\varphi = \pi/2$, the incident polarization is normal to the scattering plane. At $\theta = \pi/2$ we are measuring along the axis of the induced *magnetic* dipole moment. In this case, the radiation must be complete due to the oscillating electric dipole. This shows that the maximum strength of magnetic dipole scattering is a quarter as strong as electric dipole scattering.