Mathematical Reference for Electrodynamics

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1. RECTANGULAR COORDINATES

(x, y, z) where $x_1 = x, x_2 = y, x_3 = z, x_4 = x$, etc

1.1 Differential Operators

$$\nabla\Phi = \hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z} \quad \text{or} \quad \nabla\Phi = \sum_{i=1}^{3}\hat{\mathbf{x}}_i\frac{\partial\Phi}{\partial x_i}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{or} \quad \nabla\cdot\mathbf{A} = \sum_{i=1}^{3}\frac{\partial A_i}{\partial x_i}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad \text{or} \quad \nabla\times\mathbf{A} = \sum_{i=1}^{3}\hat{\mathbf{x}}_i\left(\frac{\partial A_{i+2}}{\partial x_{i+1}} - \frac{\partial A_{i+1}}{\partial x_{i+2}}\right) \quad \text{or}$$

$$\nabla\times\mathbf{A} = \sum_{ijk}\hat{\mathbf{x}}_i\epsilon_{ijk}\frac{\partial A_k}{\partial x_j} \quad \text{where} \quad \epsilon_{ijk} = +1 \text{ for even permutations , -1 for odd, and 0 otherwise}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \quad \text{or} \quad \nabla^2\Phi = \sum_{i=1}^{3}\frac{\partial^2\Phi}{\partial x_i^2}$$

2. CYLINDRICAL COORDINATES

(ρ, ϕ, z) = (radius, azimuth, height)

2.1 Relation to Rectangular Coordinates:

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}, \quad \tan\phi = \frac{y}{x}$$

$$z = z$$

2.2 Unit Vectors:

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\rho}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$\hat{\mathbf{x}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\rho}} - \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\rho}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

2.3 Integral Elements

$$d\mathbf{l} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$
$$d\mathbf{a}_\rho = \rho\,d\phi\,dz\,\hat{\boldsymbol{\rho}}$$
$$d\mathbf{a}_\phi = d\rho\,dz\,\hat{\boldsymbol{\phi}}$$
$$d\mathbf{a}_z = \rho\,d\rho\,d\phi\,\hat{\mathbf{z}}$$
$$dV = \rho\,d\rho\,d\phi\,dz$$

2.4 Differential Operators

$$\nabla\Phi = \hat{\boldsymbol{\rho}}\frac{\partial\Phi}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}, \qquad \nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \hat{\boldsymbol{\rho}}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right] + \hat{\mathbf{z}}\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right], \quad \nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

3. SPHERICAL COORDINATES

(r, θ, ϕ) = (radius, polar angle, azimuthal angle)

3.1 Relation to Rectangular Coordinates:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\sin\theta = \sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}, \quad \tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}, \quad \tan\phi = \frac{y}{x}$$

3.2 Unit Vectors:

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

$$\hat{\mathbf{r}} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left[x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}\right]$$
$$\hat{\boldsymbol{\theta}} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left[\frac{z\,x}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{z\,y}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} - \sqrt{x^2+y^2}\,\hat{\mathbf{z}}\right]$$
$$\hat{\boldsymbol{\phi}} = \frac{1}{\sqrt{x^2+y^2}}\left[-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}\right]$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{x}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} + \frac{x\,z}{\sqrt{x^2+y^2+z^2}}\frac{1}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\theta}} - \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} + \frac{y\,z}{\sqrt{x^2+y^2+z^2}}\frac{1}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\theta}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} - \sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\,\hat{\boldsymbol{\theta}}$$

3.3 Integral Elements

$$d\,\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$
$$d\,\mathbf{a}_r = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}, \quad d\,\mathbf{a}_\theta = r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}}, \quad d\,\mathbf{a}_\phi = r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$$
$$d\,V = r^2\sin\theta\,dr\,d\theta\,d\phi$$

3.4 Differential Operators

$$\nabla\Phi = \hat{\mathbf{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\,A_\phi\right)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,A_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\,\Phi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\Phi^2}$$

4. VECTOR IDENTITIES

$$\mathbf{A}\times\mathbf{B}=|\mathbf{A}||\mathbf{B}|\sin\theta\,\hat{\mathbf{n}}$$

$$\mathbf{A}\times\mathbf{B}=\hat{\mathbf{x}}(A_y B_z-A_z B_y)+\hat{\mathbf{y}}(A_z B_x-A_x B_z)+\hat{\mathbf{z}}(A_x B_y-A_y B_x)$$

$$\mathbf{A}\times\mathbf{B}=\sum_{i=1}^{3}\hat{\mathbf{x}}_i\left(A_{i+1}B_{i+2}-A_{i+2}B_{i+1}\right)$$

$$\mathbf{A}\times\mathbf{B}=\sum_{ijk}\hat{\mathbf{x}}_i\epsilon_{ijk}A_j B_k$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=-4\pi\delta(\mathbf{x}-\mathbf{x}')$$

$$\nabla\times\nabla\Phi=0$$

$$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\int_V\nabla\cdot\mathbf{A}\,d^3x=\int_S\mathbf{A}\cdot d\,\mathbf{a}$$

$$\int_V(\Phi\nabla^2\Psi-\Psi\nabla^2\Phi)\,d^3x=\int_S(\Phi\nabla\Psi-\Psi\nabla\Phi)\cdot d\,\mathbf{a}$$

$$\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}||\mathbf{B}|\cos\theta$$

$$\mathbf{A}\cdot\mathbf{B}=A_x B_x+A_y B_y+A_z B_z$$

$$\mathbf{A}\cdot\mathbf{B}=\sum_{i=1}^{3}A_i B_i$$

$$\mathbf{A}\cdot\mathbf{B}=\sum_{i=1}^{3}(\hat{\mathbf{x}}_i\cdot\mathbf{A})(\hat{\mathbf{x}}_i\cdot\mathbf{B})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=-\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)$$

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=\nabla'^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)$$

$$\nabla\cdot(\nabla\times\mathbf{A})=0$$

$$\nabla(\psi\phi)=\phi\nabla\psi+\psi\nabla\phi$$

$$\nabla\cdot(\Phi\mathbf{A})=\mathbf{A}\cdot\nabla\Phi+\Phi\nabla\cdot\mathbf{A}$$

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

$$\int_S(\nabla\times\mathbf{A})\cdot d\,\mathbf{a}=\oint_C\mathbf{A}\cdot d\,\mathbf{l}$$

5. ORTHOGONAL FUNCTIONS (*m*, *n*, and *l* are integers)

5.1 Orthogonality Statements

$$\int_0^a\cos\left(\frac{2\pi mx}{a}\right)\cos\left(\frac{2\pi nx}{a}\right)dx=\frac{a}{2}\delta_{mn}$$

$$\int_0^a\sin\left(\frac{2\pi mx}{a}\right)\sin\left(\frac{2\pi nx}{a}\right)dx=\frac{a}{2}\delta_{mn}$$

$$\int_0^a e^{i2\pi(m-n)x/a}dx=a\,\delta_{mn}$$

$$\int_{-\infty}^{\infty}e^{i(k-k')x}dx=2\pi\delta(k-k')$$

$$\int_{-1}^{1}P_{l'}(x)P_l(x)\,dx=\frac{2}{2l+1}\delta_{ll'}$$

$$\int_{-1}^{1}P_{l'}^m(x)P_l^m(x)\,dx=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}$$

$$\int_0^a\rho J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)d\rho=\frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2\delta_{n'n}$$

$$\int_0^{2\pi}\cos(mx)\cos(nx)\,dx=\pi\delta_{mn}$$

$$\int_0^{2\pi}\sin(mx)\sin(nx)\,dx=\pi\delta_{mn}$$

$$\int_0^{2\pi}e^{i(m-n)x}dx=2\pi\delta_{mn}$$

$$\int_{-\infty}^{\infty}e^{i(x-x')k}d\,k=2\pi\delta(x-x')$$

$$\int_0^{\pi}P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta=\frac{2}{2l+1}\delta_{ll'}$$

$$\int_0^{2\pi}\int_0^{\pi}Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi)\sin\theta\,d\theta\,d\phi=\delta_{ll'}\delta_{mm'}$$

$$\int_0^{\infty}x J_m(kx)J_m(k'x)\,dx=\frac{1}{k}\delta(k'-k)$$

5.2 Legendre Polynomials

$$P_0(x)=1\,,\quad P_1(x)=x\,,\quad P_2(x)=\frac{3}{2}x^2-\frac{1}{2}\,,\quad P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x\,,\quad P_4(x)=\frac{35}{8}x^4-\frac{30}{8}x^2+\frac{3}{8}$$

$$P_l(-x)=(-1)^l P_l(x)\,,\quad P_l(1)=1$$

$$(l+1)P_{l+1}(x)-(2l+1)x\,P_l(x)+l\,P_{l-1}(x)=0\,,\quad P_l(x)=\frac{1}{2l+1}\frac{d}{dx}\left[P_{l+1}(x)-P_{l-1}(x)\right]$$

5.3 Associated Legendre Functions

$$P_l^0(x)=P_l(x)\,,\quad P_1^1=-\sqrt{1-x^2}\,,\quad P_2^1(x)=-3x\sqrt{1-x^2}\,,\quad P_2^2=3-3x^2\,,\quad P_3^1(x)=-\frac{3}{2}(5x^2-1)\sqrt{1-x^2}$$

$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

5.4 Spherical Harmonics

$$Y_{l,-m}=(-1)^m Y^*_{lm}$$

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

$$Y_{00}=\frac{1}{\sqrt{4\pi}}\qquad Y_{10}=\sqrt{\frac{3}{4\pi}}\cos\theta\qquad Y_{20}=\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right)\qquad Y_{30}=\sqrt{\frac{7}{16\pi}}\left(5\cos^3\theta-3\cos\theta\right)$$

$$Y_{11}=-\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{i\phi}\qquad Y_{21}=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{i\phi}\qquad Y_{31}=-\sqrt{\frac{21}{64\pi}}\sin\theta(5\cos^2\theta-1)\,e^{i\phi}$$

$$Y_{22}=\sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{i2\phi}\qquad Y_{32}=\sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta\,e^{i2\phi}$$

$$Y_{33}=-\sqrt{\frac{35}{64\pi}}\sin^3\theta\,e^{i3\phi}$$

5.5 Bessel Functions

$$J_0(0)=1\,,\quad J_n(0)=0\ \text{for}\ n\neq 0\,,\quad N_m(0)=-\infty\,,\quad J_0'(x)=-J_1(x)\,,\quad J_{-m}(x)=(-1)^m J_m(x)$$

$$J_{n+1}=\frac{2n}{x}J_n(x)-J_{n-1}(x)$$

$$J_n'(x)=\frac{1}{2}J_{n-1}(x)+\frac{1}{2}J_{n+1}(x)$$

$$\int x\,J_0(x)\,dx=x\,J_1(x)$$

$$\int J_1(x)\,dx=-J_0(x)$$

$$\int_0^\infty e^{-ax}J_0(bx)\,dx=\frac{1}{\sqrt{a^2+b^2}}$$

$$\int_0^\infty \cos(ax)\,J_0(bx)\,dx=\frac{1}{\sqrt{a^2-b^2}}\quad \text{if } a>b$$

$$\int_0^\infty J_n(bx)\,dx=\frac{1}{b}\ \text{for}\ n>-1$$

$$J_n(z)=\frac{1}{2\pi i^n}\int_0^{2\pi}e^{i(z\cos\theta+n\theta)}\,d\theta$$

$$N_m(x)=\frac{J_m(x)\cos(m\pi)-J_{-m}(x)}{\sin(m\pi)}$$

$$N_{-m}(x)=(-1)^m N_m(x)$$

$$H_m^{(1)}(x)=J_m(x)+i\,N_m(x)$$

$$H_m^{(2)}(x)=J_m(x)-i\,N_m(x)$$

$$I_m(x)=i^{-m}J_m(ix)$$

$$K_m(x)=\frac{\pi}{2}i^{m+1}H_m^{(1)}(ix)$$

6. COMPLEX NUMBERS

All expressions with Arg(z) have an implicit additive factor $2\pi m$ where m = 0,1,2...

$$e^{i\pi m}=(-1)^m \text{ where } m=0,1,2...$$
$$e^{i\pi m/2}=i^m \text{ where } m=0,1,2...$$
$$e^{i\theta}=\cos\theta+i\sin\theta$$

$$z=\Re(z)+i\,\Im(z)$$
$$z=|z|\,e^{i\,Arg(z)}$$

$$z^*=\Re(z)-i\,\Im(z)$$
$$z^*=|z|e^{-i\,Arg(z)}$$

$$\Re(z)=\frac{z+z^*}{2}$$
$$\Re(z)=|z|\cos(Arg(z))$$

$$\Im(z)=\frac{z-z^*}{2i}$$
$$\Im(z)=|z|\sin(Arg(z))$$

$$|z|=\sqrt{z\,z^*}$$
$$|z|=\sqrt{(\Re(z))^2+(\Im(z))^2}$$

$$Arg(z)=\sin^{-1}\left(\frac{\Im(z)}{|z|}\right)$$
$$Arg(z)=\cos^{-1}\left(\frac{\Re(z)}{|z|}\right)$$
$$Arg(z)=\tan^{-1}\left(\frac{\Im(z)}{\Re(z)}\right)$$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$
$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$\tan\theta=-i\,\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$$

$$\sinh x=\frac{e^x-e^{-x}}{2}$$
$$\cosh x=\frac{e^x+e^{-x}}{2}$$
$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$$\sinh^{-1}z=\ln(z+\sqrt{z^2+1})$$
$$\cosh^{-1}z=\ln(z+\sqrt{z^2-1})$$
$$\tanh^{-1}z=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\sin^{-1}z=-i\ln(i\,z+\sqrt{1-z^2})$$
$$\cos^{-1}z=\pm i\ln(z+\sqrt{z^2-1})$$
$$\tan^{-1}z=\frac{i}{2}\ln\left(\frac{i-z}{i+z}\right)$$

$$\sin(i\,z)=i\sinh(z)$$
$$\cos(i\,z)=\cosh(z)$$
$$\tan(i\,z)=i\tanh(z)$$

$$\sin^{-1}(i\,z)=i\sinh^{-1}(z)$$
$$\cos^{-1}(z)=\pm i\cosh^{-1}(z)$$
$$\tan^{-1}(i\,z)=i\tanh^{-1}(z)$$

$$z^n=|z|^n[\cos(n\,Arg(z))+i\sin(n\,Arg(z))]$$

$$\sqrt{z}=\frac{1}{\sqrt{2}}[\sqrt{|z|+\Re(z)}+sgn(\Im(z))\,i\sqrt{|z|-\Re(z)}]$$

$$\ln(z)=\ln(|z|)+i\,Arg(z)$$

$$\tan^{-1}(z)=\left[\frac{1}{2}\tan^{-1}\left(\frac{2\Re(z)}{1-|z|^2}\right)\right]+i\left[\frac{1}{4}\ln\left(\frac{1+2\Im(z)+|z|^2}{1-2\Im(z)+|z|^2}\right)\right]$$

$$\Im(z_1)\Im(z_2)=\frac{1}{2}\Re[-z_1 z_2+z_1^* z_2]$$

7. TRIGONOMETRY IDENTITIES

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

$$\sin(2A)=2\sin A\cos A$$

$$\cos(2A)=\cos^2 A-\sin^2 A$$

$$\sin(2A)=2\sin A\sqrt{1-\sin^2 A}$$

$$\cos(2A)=2\cos^2 A-1$$

$$\sin(2A)=2\cos A\sqrt{1-\cos^2 A}$$

$$\cos(2A)=1-2\sin^2 A$$

$$\sin\left(\frac{A}{2}\right)=\pm\sqrt{\frac{1-\cos A}{2}}$$

+ if $A/2$ in quad. I or II
- if $A/2$ in quad. III or IV

$$\cos\left(\frac{A}{2}\right)=\pm\sqrt{\frac{1+\cos A}{2}}$$

+ if $A/2$ in quad. I or IV
- if $A/2$ in quad. II or III

$$\sin(-\theta)=-\sin\theta$$

$$\cos(-\theta)=\cos\theta$$

$$\sin\left(\frac{\pi}{2}\pm\theta\right)=\cos\theta$$

$$\cos\left(\frac{\pi}{2}\pm\theta\right)=\mp\sin\theta$$

$$\sin(\theta\pm\pi)=-\sin\theta$$

$$\cos(\theta\pm\pi)=-\cos\theta$$

$$\sinh(-\theta)=-\sinh\theta$$

$$\cosh(-\theta)=\cosh\theta$$

$$\cos^2 A+\sin^2 A=1$$

$$\cosh^2 A-\sinh^2 A=1$$

$$\cos^{-1}x+\sin^{-1}x=\pi/2$$

$$\cosh^{-1}(1/x)=1/\cosh^{-1}x$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$$\tan^{-1}a\pm\tan^{-1}b=\tan^{-1}\left(\frac{a\pm b}{1\mp a\,b}\right)$$

$$\tan\left(\frac{A}{2}\right)=\frac{1-\cos A}{\sin A}$$

$$\tan\left(\frac{A}{2}\right)=\frac{\sin A}{1+\cos A}$$

$$\tan(2A)=\frac{2\tan A}{1-\tan^2 A}$$

$$\tan(2A)=\frac{\cos A\sqrt{1-\cos^2 A}}{\cos^2 A-1/2}$$

$$\tan A=\frac{\sin A}{\cos A}$$

$$\tanh A=\frac{\sinh A}{\cosh A}$$

8. OTHER SPECIAL FUNCTIONS

8.1 Logarithm and Exponentiation

$\ln(ab)=\ln a+\ln b$ $\ln(a/b)=\ln a-\ln b$ $\ln(a^b)=b\ln a$ $e^{\ln a}=a$

$(a^p)(a^q)=a^{p+q}$ $(a^p)/(a^q)=a^{p-q}$ $a^{-p}=1/a^p$ $(a^p)^q=a^{pq}$

8.2 Gamma Function and Factorials

$\Gamma(0)=\infty$, $\Gamma(1/2)=\sqrt{\pi}$, $\Gamma(1)=1$, $\Gamma(2)=1$, $\Gamma(n+1)=n\Gamma(n)$

$\Gamma(n)=(n-1)!$ if n is a positive integer

$n!=1\times2\times3\times\ldots\times n$ where $0!=1$

$n!!=n(n-2)(n-4)\ldots$ where $0!!=(-1)!!=1$

8.3 Dirac Delta

$\delta(x-a)=0$ if $x\neq a$

$$\int f(x)\delta(x-a)dx=f(a)$$

$$\delta(-x)=\delta(x)$$

$$\delta(x-a)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-a)k}\,dk$$

$$\delta(ax)=\frac{1}{|a|}\delta(x)$$

$$\delta(x^2-a^2)=\frac{1}{2a}(\delta(x+a)+\delta(x-a))$$

$$\delta(f(x))=\sum_i \frac{\delta(x-x_i)}{\left|\left(\frac{df}{dx}\right)_{x=x_i}\right|}$$ where x_i are the points where $f(x) = 0$

$\delta(\cos\theta-\cos\theta')=\delta(\theta-\theta')/\sin\theta$ for $0<\theta<\pi$

$\delta^{(3)}(\mathbf{x}-\mathbf{x}')=\delta(u-u')\delta(v-v')\delta(w-w')\,U\,V\,W$ where the length elements are $(du/U, dv/V, dw/W)$

$$\delta_{\text{rect}}(\mathbf{x}-\mathbf{x}')=\delta(x-x')\delta(y-y')\delta(z-z')$$

$$\delta_{\text{sphere}}(\mathbf{x}-\mathbf{x}')=(\delta(r-r'))\left(\frac{\delta(\theta-\theta')}{r}\right)\left(\frac{\delta(\phi-\phi')}{r\sin\theta}\right)$$

$$\delta_{\text{cyl}}(\mathbf{x}-\mathbf{x}')=(\delta(\rho-\rho'))\left(\frac{\delta(\phi-\phi')}{\rho}\right)(\delta(z-z'))$$

9. EXPANSIONS

9.1 Taylor Series General Form

$$f(x)=f(a)+(x-a)\left[\frac{df}{dx}\right]_{x=a}+\frac{1}{2}(x-a)^2\left[\frac{\partial^2 f}{\partial x^2}\right]_{x=a}+\ldots$$

$$f(\mathbf{x})=f(\mathbf{a})+\sum_i (x_i-a_i)\left[\frac{\partial f}{\partial x_i}\right]_{\mathbf{x}=\mathbf{a}}+\frac{1}{2}\sum_{i,j}(x_i-a_i)(x_j-a_j)\left[\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}f\right]_{\mathbf{x}=\mathbf{a}}+\ldots$$

9.2 Special Taylor Series

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots$$

$$\sin^{-1}x=\sum_{n=0}^{\infty}\frac{(2n-1)!!\,x^{2n+1}}{(2n)!!(2n+1)}$$

$$\cos^{-1}x=\frac{\pi}{2}-\sum_{n=0}^{\infty}\frac{(2n-1)!!\,x^{2n+1}}{(2n)!!(2n+1)}$$

$$\tan^{-1}(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

$$\sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

$$\tanh x=x-\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots$$

$$\sinh^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!\,x^{2n+1}}{(2n)!!(2n+1)}$$

$$\cosh^{-1}x=\ln(2x)-\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!\,2n\,x^{2n}}$$

$$\tanh^{-1}(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

$$\frac{x}{x-1}=\sum_{n=0}^{\infty}x^{-n}$$

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

$$\frac{1}{1+x}=(-1)^n\sum_{n=0}^{\infty}x^n$$

$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$$

$$1/\sqrt{1+x}=1-\frac{1}{2}x+\frac{3}{8}x^2+\ldots$$

9.3 Unit Point Potential Expansions

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$ (Cartesian coordinates)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{\sqrt{r^2+r'^2-2rr'(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'))}}$$ (Spherical coordinates)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}}$$ (Cylindrical coordinates)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$ (Spherical coordinates series)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty}dk\,e^{im(\phi-\phi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)$$ (Cylindrical coordinates series)

10. INTEGRALS

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}$$

$$\int \frac{1}{x}\,dx=\ln x$$

$$\int \frac{1}{a^2+x^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2-x^2}\,dx=-\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx=\sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx=-\cos^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx=\cosh^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{a^2-x^2}\,dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{a^2+x^2}\,dx=\frac{1}{2}x\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int e^{ax}\,dx=\frac{e^{ax}}{a}$$

$$\int x\,e^{ax}\,dx=\frac{e^{ax}}{a}\left(x-\frac{1}{a}\right)$$

$$\int \sin(ax)\,dx=-\frac{\cos(ax)}{a}$$

$$\int \cos(ax)\,dx=\frac{\sin(ax)}{a}$$

$$\int x\sin(ax)\,dx=\frac{\sin(ax)}{a^2}-\frac{x\cos(ax)}{a}$$

$$\int x\cos(ax)\,dx=\frac{\cos(ax)}{a^2}+\frac{x\sin(ax)}{a}$$

$$\int \sinh(ax)\,dx=\frac{1}{a}\cosh(ax)$$

$$\int \cosh(ax)\,dx=\frac{1}{a}\sinh(ax)$$

$$\int \frac{1}{\sin^2(ax)}\,dx=-\frac{1}{a\tan(ax)}$$

$$\int \frac{1}{\cos^2(ax)}\,dx=\frac{1}{a}\tan(ax)$$

$$\int \frac{1}{\sinh^2(ax)}\,dx=-\frac{1}{a\tanh(ax)}$$

$$\int \frac{1}{\cosh^2(ax)}\,dx=\frac{1}{a}\tanh(ax)$$

$$\int \tan(ax)\,dx=-\frac{1}{a}\ln(\cos(ax))$$

$$\int \tanh(ax)\,dx=\frac{1}{a}\ln(\cosh(ax))$$

$$\int_0^\infty x^n e^{-ax}\,dx=\frac{\Gamma(n+1)}{a^{n+1}}$$

$$\int_0^\infty e^{-ax^2}=\sqrt{\frac{\pi}{4a}}$$

11. DERIVATIVES

$$\frac{d}{dx}(u^n)=n\,u^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}\ln u=\frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}e^u=e^u\frac{du}{dx}$$

$$\frac{d}{dx}\sin u=\cos u\frac{du}{dx}$$

$$\frac{d}{dx}\cos u=-\sin u\frac{du}{dx}$$

$$\frac{d}{dx}\tan u=\frac{1}{\cos^2 u}\frac{du}{dx}$$

$$\frac{d}{dx}\sinh u=\cosh u\frac{du}{dx}$$

$$\frac{d}{dx}\cosh u=\sinh u\frac{du}{dx}$$

$$\frac{d}{dx}\tanh u=\frac{1}{\cosh^2 u}\frac{du}{dx}$$

$$\frac{d}{dx}\sin^{-1}u=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\cos^{-1}u=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\sinh^{-1}u=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\cosh^{-1}u=-\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}\tanh^{-1}u=\frac{1}{1-u^2}\frac{du}{dx}$$