Magnetostatic Equations Summary

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1. Definitions

$\mu_0\mathbf{H}$: The *applied* magnetic field plus interactions.
$\mu_0\mathbf{M}$: The *material response* magnetic field.
B: The *total* magnetic field.

J: The *applied* current density.
$\mathbf{J}_M$: The *material* current density
$\mathbf{J}_{\text{total}}$: The *total* current density

2. Fundamental Equations to Understand and Memorize

$\mathbf{B}=\mu_0\mathbf{H}+\mu_0\mathbf{M}$	(total field = applied field + material response field)
$\mathbf{J}_{\text{total}}=\mathbf{J}+\mathbf{J}_M$	(total current density = applied current density + material current density)

$\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}}$	(total current gives rise to curling total field)
$\nabla\times\mathbf{H}=\mathbf{J}$	(applied current gives rise to curling applied field)
$\nabla\times\mathbf{M}=\mathbf{J}_M$	(material current gives rise to curling material response field)
$\nabla\times\nabla\Phi=0$	(math identity: if a vector field has no curl, it is the gradient of a scalar potential)

$\nabla\cdot\mathbf{B}=0$	(total magnetic field lines are not created or destroyed)
$\nabla\cdot\mathbf{H}=\rho_M$	(positive effective magnetic charge creates **H** field lines, negative destroys)
$\nabla\cdot\mathbf{M}=-\rho_M$	(negative effective magnetic charge creates **H** field lines, positive destroys)
$\nabla\cdot(\nabla\times\mathbf{A})=0$	(math identity: if a vector field has zero divergence, it is the curl of a vector field)

$\mathbf{B}=\mu\mathbf{H}$	(linear magnetic material have the applied and total fields linked by a constant)

3. Derived Equations: Potentials

$$\begin{tikzcd}
\boxed{\nabla\times\mathbf{H}=\mathbf{J}} \arrow[dr] & \boxed{\mathbf{J}=0} \arrow[d] & \boxed{\nabla\times\nabla\Phi=0} \arrow[dl] \\
& \boxed{\mathbf{H}=-\nabla\Phi_M} \arrow[d] & \boxed{\nabla\cdot\mathbf{H}=\rho_M} \arrow[dl] \\
& \boxed{\nabla^2\Phi_M=-\rho_M} &
\end{tikzcd}$$

(true in regions with
no applied current)

$$\begin{tikzcd}
\boxed{\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}}} \arrow[dr] & \boxed{\mathbf{J}_{\text{total}}=0} \arrow[d] & \boxed{\nabla\times\nabla\Phi=0} \arrow[dl] \\
& \boxed{\mathbf{B}=-\nabla\Psi_M} \arrow[d] & \boxed{\nabla\cdot\mathbf{B}=0} \arrow[dl] \\
& \boxed{\nabla^2\Psi_M=0} &
\end{tikzcd}$$

(true in regions with
no total current)

$$\begin{tikzcd}
\boxed{\nabla\cdot\mathbf{B}=0} \arrow[dr] & & \boxed{\nabla\cdot(\nabla\times\mathbf{A})=0} \arrow[dl] \\
& \boxed{\mathbf{B}=\nabla\times\mathbf{A}} \arrow[d] & \boxed{\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}}} \arrow[dl] \\
& \boxed{\nabla^2\mathbf{A}=-\mu_0\mathbf{J}_{\text{total}}} &
\end{tikzcd}$$

(always true)

$$\begin{tikzcd}
\boxed{\nabla^2\mathbf{A}=-\mu_0\mathbf{J}_{\text{total}}} \arrow[dr] & & \boxed{\mathbf{J}=0} \arrow[dl] \\
& \boxed{\nabla^2\mathbf{A}=-\mu_0\mathbf{J}_M} \arrow[d] & \boxed{\mathbf{J}_M=0} \arrow[dl] \\
& \boxed{\nabla^2\mathbf{A}=0} &
\end{tikzcd}$$

(true in regions with
no total current)

5. Derived Equations: Boundary Conditions

$$\begin{tikzcd}
\boxed{\nabla\cdot\mathbf{B}=0} \arrow[d] & \boxed{\nabla\cdot\mathbf{H}=\rho_M} \arrow[d] & \boxed{\nabla\cdot\mathbf{M}=-\rho_M} \arrow[d] \\
\boxed{(\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n}=0} & \boxed{(\mathbf{H}_2-\mathbf{H}_1)\cdot\mathbf{n}=\sigma_M} & \boxed{(\mathbf{M}_2-\mathbf{M}_1)\cdot\mathbf{n}=-\sigma_M}
\end{tikzcd}$$

(using Gaussian
pillbox surface)

$$\begin{tikzcd}
\boxed{\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}}} \arrow[d] & \boxed{\nabla\times\mathbf{H}=\mathbf{J}} \arrow[d] & \boxed{\nabla\times\mathbf{M}=\mathbf{J}_M} \arrow[d] \\
\boxed{\mathbf{n}\times(\mathbf{B}_2-\mathbf{B}_1)=\mu_0\mathbf{K}_{\text{total}}} & \boxed{\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}} & \boxed{\mathbf{n}\times(\mathbf{M}_2-\mathbf{M}_1)=\mathbf{K}_M}
\end{tikzcd}$$

(using Amperian
loop)

4. Derived Equations: Linear Materials

$$
\begin{tikzcd}
\boxed{\mathbf{B}=\mu\mathbf{H}} \arrow[d] \arrow[dr] & \boxed{\mathbf{B}=\mu_0\mathbf{H}+\mu_0\mathbf{M}} \arrow[dl] \arrow[d] \\
\boxed{\mathbf{B}=\left(\dfrac{\mu\mu_0}{\mu-\mu_0}\right)\mathbf{M}} & \boxed{\mathbf{H}=\left(\dfrac{\mu_0}{\mu-\mu_0}\right)\mathbf{M}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{\mathbf{B}=\mu\mathbf{H}} \arrow[dr] & & \boxed{\nabla^2\mathbf{A}=-\mu_0\mathbf{J}_{\text{total}}} \arrow[dl] \\
& \boxed{\nabla^2\mathbf{A}=-\mu\mathbf{J}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{\mathbf{B}=\mu\mathbf{H}} \arrow[dr] & & \boxed{(\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n}=0} \arrow[dl] \\
& \boxed{(\mu_2\mathbf{H}_2-\mu_1\mathbf{H}_1)\cdot\mathbf{n}=0} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{\mathbf{B}=\mu\mathbf{H}} \arrow[dr] & & \boxed{\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}} \arrow[dl] \\
& \boxed{\mathbf{n}\times\left(\dfrac{1}{\mu_2}\mathbf{B}_2-\dfrac{1}{\mu_1}\mathbf{B}_1\right)=\mathbf{K}} &
\end{tikzcd}
$$

In regions of linear *uniform* material and $\mathbf{J} = 0$, we can show $\mathbf{J}_M = 0$, and therefore $\mathbf{J}_{\text{total}} = 0$.

In free space (no magnetic materials), $\mathbf{M} = 0$, $\mathbf{J}_M = 0$, and therefore $\mathbf{B} = \mu_0\mathbf{H}$ and $\mathbf{J}_{\text{total}} = \mathbf{J}$.