Lecture 4 Supplemental Notes, Electromagnetic Theory I

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We will go step by step through a simple problem in order to fully understand the procedure for solving any Laplace equation problem.

Consider a two-dimensional box with one corner at the origin and the other corner at $(x, y) = (a, b)$. All sides of the box have a potential of zero except for the side at $y = b$ which is held at the potential $V_0(x/a)^2$. We wish to find the potential everywhere inside the box.

***Step 1*: Write down the Boundary Conditions**

$$\Phi(x=0)=0$$

$$\Phi(x=a)=0$$

$$\Phi(y=0)=0$$

$$\Phi(y=b)=V(x) \text{ where } V(x)=V_0\left(\frac{x}{a}\right)^2$$

***Step 2*: Find the General Solution to the Laplace Equation**

In two-dimensional rectangular coordinates, the Laplace equation is:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0$$

Try a solution of the form: $\Phi=X(x)Y(y)$

$$\frac{\partial^2[X(x)Y(y)]}{\partial x^2}+\frac{\partial^2[X(x)Y(y)]}{\partial y^2}=0$$

The partial derivatives become total derivatives because they operate on functions of only one variable:

$$Y(y)\frac{d^2X(x)}{dx^2}+X(x)\frac{d^2Y(y)}{dy^2}=0$$

Divide by $X(x)$ and $Y(y)$ to find:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=0$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$$

The two terms are independent and variable, so they must be related by a constant.

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-\alpha^2 \quad \text{where} \quad \alpha^2=\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$$

Put each equation in a more standard form:

$$\frac{d^2X(x)}{dx^2}=-\alpha^2X(x) \quad \text{where} \quad \frac{d^2Y(y)}{dy^2}=\alpha^2Y(y)$$

TIP: Always handle the homogenous ($\alpha = 0$) and inhomogenous cases ($\alpha \neq 0$) separately!

For the homogenous case ($\alpha = 0$), these equations become:

$$\frac{d^2X(x)}{dx^2}=0 \quad \text{where} \quad \frac{d^2Y(y)}{dy^2}=0$$

with the solutions:

$$X(x)=A_0+B_0x \quad \text{and} \quad Y(y)=C_0+D_0x$$

so that this particular solution becomes:

$$\Phi=X(x)Y(y)$$

$$\Phi=(A_0+B_0x)(C_0+D_0y)$$

For the inhomogenous case ($\alpha \neq 0$), these equations stay as:

$$\frac{d^2X(x)}{dx^2}=-\alpha^2X(x) \quad \text{where} \quad \frac{d^2Y(y)}{dy^2}=\alpha^2Y(y)$$

The solutions to these are exponentials:

$$X(x)=A_\alpha e^{i\alpha x}+B_\alpha e^{-i\alpha x} \quad \text{and} \quad Y(y)=C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y}$$

So that the particular solution is:

$$\Phi=X(x)Y(y)$$

$$\Phi=(A_\alpha e^{i\alpha x}+B_\alpha e^{-i\alpha x})(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

Note that I could have just as easily set the x differential equation equal to $+\alpha^2$ and the y differential equation equal to $-\alpha^2$:

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}=\alpha^2 \quad \text{where} \quad \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}=-\alpha^2$$

in which case the particular solution would be:

$$\Phi=(A_\alpha e^{\alpha x}+B_\alpha e^{-\alpha x})(C_\alpha e^{i\alpha y}+D_\alpha e^{-i\alpha y}) \quad \text{instead of} \quad \Phi=(A_\alpha e^{i\alpha x}+B_\alpha e^{-i\alpha x})(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

A little thinking shows that these solutions are identical. Because α is arbitrary at this point, we can make the transformation $\alpha \rightarrow i\alpha$ which turns the solution on the left to the one on the right. We should pick one of these forms and stick with it.
The general solution is the sum of all particular solutions:

$$\Phi=(A_0+B_0 x)(C_0+D_0 y)+\sum_{\alpha\neq 0}(A_\alpha e^{i\alpha x}+B_\alpha e^{-i\alpha x})(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

***Step 3*: Apply the Boundary Conditions**

TIP: Apply the simplest boundary conditions first!

Apply the first boundary condition at $x = 0$:

$$\Phi(x=0)=0$$

$$0=(A_0+B_0(0))(C_0+D_0 y)+\sum_{\alpha\neq 0}(A_\alpha e^{i\alpha(0)}+B_\alpha e^{-i\alpha(0)})(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

$$0=A_0(C_0+D_0 y)+\sum_{\alpha\neq 0}(A_\alpha+B_\alpha)(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

Since every particular solution is orthogonal to every other particular solution, its coefficient must vanish separately, so that:

$$0=A_0(C_0+D_0 y) \quad \text{and} \quad 0=(A_\alpha+B_\alpha)(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

Since these equations must be true for all values of y, the coefficients out front must be zero.

$$0=A_0 \quad \text{and} \quad 0=A_\alpha+B_\alpha$$

so that:

$$\boxed{0=A_0} \quad \text{and} \quad \boxed{B_\alpha=-A_\alpha}$$

This is the one piece of information that this boundary condition gives us (or, more correctly, the one piece information for each particular solution).

TIP: After each boundary condition is applied and another piece of information is found, write down the overall solution so far with the value of the coefficient inserted. Then use this form when applying subsequent boundary conditions. This will save you a lot of writing!

With this information applied, the overall solution to the problem now becomes:

$$\Phi=(A_0+B_0 x)(C_0+D_0 y)+\sum_{\alpha\neq 0}(A_\alpha e^{i\alpha x}+B_\alpha e^{-i\alpha x})(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

$$\Phi=B_0 x(C_0+D_0 y)+\sum_{\alpha\neq 0}(A_\alpha e^{i\alpha x}-A_\alpha e^{-i\alpha x})(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

$$\Phi=B_0 x(C_0+D_0 y)+\sum_{\alpha\neq 0}A_\alpha(e^{i\alpha x}-e^{-i\alpha x})(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

Use the relation $e^{i\alpha x}-e^{-i\alpha x}=2i\sin(\alpha x)$ and suck the $2i$ and the coefficient A_α into C_α and D_α to find:

$$\boxed{\Phi=B_0 x(C_0+D_0 y)+\sum_{\alpha\neq 0}\sin(\alpha x)(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})}$$

This is our overall solution so far. When we apply the next boundary condition, we should use this form to save ourselves some effort.

TIP: When applying the information gained from a boundary condition, apply it to the overall solution that is valid everywhere and not the expression for the potential just on the boundary.

Now apply the boundary condition at $x = a$:

$$\Phi(x=a)=0$$

$$0=B_0 a(C_0+D_0 y)+\sum_{\alpha\neq 0}\sin(\alpha a)(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

Again, the coefficients must match up separately because the functions are orthogonal:

$$0=B_0 a(C_0+D_0 y)\quad\text{and}\quad 0=\sin(\alpha a)(C_\alpha e^{\alpha y}+D_\alpha e^{-\alpha y})$$

Again, these equations must be true for all y, so the coefficients must be zero:

$$0=B_0\quad\text{and}\quad 0=\sin(\alpha a)$$

The sine function is zero for every integer multiple of pi so that:

$$0 = B_0 \quad \text{and} \quad \alpha a = n\pi \quad \text{where } n = 1, 2, 3...$$

$$\boxed{0 = B_0} \quad \text{and} \quad \boxed{\alpha = \frac{n\pi}{a}} \quad \text{where } n = 1, 2, 3...$$

This is the piece of information that the $x = a$ boundary condition gives us. Writing down the overall solution so far with this information inserted, we have:

$$\boxed{\Phi = \sum_{n=1,2,3...} \sin\left(\frac{n\pi x}{a}\right)\left(C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}\right)} \quad \text{where } n = 1, 2, 3...$$

Note that the subscripts on the undetermined coefficients and the summation symbol were changed from α to n to show that we have found α and that we now know that different terms in the series vary according to the integer n. By plugging in $x = 0$ and $x = a$, we can check that this equation obeys the boundary conditions that we have already applied, which it does.
Now apply the boundary condition at $y = 0$:

$$\Phi(y=0) = 0$$

$$0 = \sum_{n=1,2,3...} \sin\left(\frac{n\pi x}{a}\right)\left(C_n e^{n\pi(0)/a} + D_n e^{-n\pi(0)/a}\right)$$

$$0 = \sum_{n=1,2,3...} \sin\left(\frac{n\pi x}{a}\right)\left(C_n + D_n\right)$$

Again, the functions are orthogonal, so they must vanish separately:

$$0 = \sin\left(\frac{n\pi x}{a}\right)\left(C_n + D_n\right)$$

Since this must be valid for all x, the coefficient must vanish:

$$0 = C_n + D_n$$

$$\boxed{D_n = -C_n}$$

This is the next piece of information. Insert it in our overall solution so far to find:

$$\Phi = \sum_{n=1,2,3...} \sin\left(\frac{n\pi x}{a}\right)\left(C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}\right)$$

$$\Phi = \sum_{n=1,2,3...} \sin\left(\frac{n\pi x}{a}\right)\left(C_n e^{n\pi y/a} - C_n e^{-n\pi y/a}\right)$$

$$\Phi = \sum_{n=1,2,3...} C_n \sin\left(\frac{n\pi x}{a}\right)\left(e^{n\pi y/a} - e^{-n\pi y/a}\right)$$

Use $e^A - e^{-A} = 2\sinh A$ and suck the 2 into the C_n to find:

$$\boxed{\Phi = \sum_{n=1,2,3...} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh(n\pi y/a)}$$

At this point we have only one set of undermined coefficients left, C_n, and one boundary condition left, at $y = b$. Apply this last boundary condition:

$$\Phi(y=b) = V(x) \text{ where } V(x) = V_0\left(\frac{x}{a}\right)^2$$

TIP: For the complicated boundary condition, use the general form $V(x)$ as long as possible. Only insert the explicit expression for this boundary condition at the end when it is already sitting inside an integral. This will save you some writing and will keep you from making the mistake of splitting up a single boundary into smaller pieces which you treat as independent!

$$V(x) = \sum_{n=1,2,3...} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh(n\pi b/a)$$

Now at this point we need so solve for C_n, but it is stuck in a series. Also, the boundary condition is not so simple that we can match up coefficients in the series. We need to get C_n out of the summation. The thing that can do this for us is a Kronecker delta, because a Kronecker delta is zero for all arguments except when its arguments match:

Kronecker Delta definition: $\delta_{n,n'} = \begin{Bmatrix} 0 \text{ if } n \neq n' \\ 1 \text{ if } n = n' \end{Bmatrix}$

In general $\sum_n C_n \delta_{n,n'} = C_{n'}$ because the Kronecker delta is zero for all terms in the series except the one special one, and therefore makes all other terms go away. The Kronecker delta therefore is said to "collapse the summation symbol", or to "pick out one representative term from the series". This is exactly what we want in order to solve for C_n. Therefore, we need to create the Kronecker delta. We can do this by realizing that by integrating over orthogonal functions we get the Kronecker delta. For the sine functions:

$$\boxed{\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) dx = \frac{a}{2}\delta_{n,n'}} \qquad \textit{Orthogonality statement of sine functions}$$

We could prove this by simply doing the integral for both cases, $n' = n$ and $n' \neq n$. Note that there are some constants which we can't forget that result when we do the integral, in this case $a/2$.
If we can create the integral shown above so that it appears in the last boundary condition statement, we can create the Kronecker delta and therefore collapse the series. We do this by multiplying and integrating both sides of the solution:

$$V(x) = \sum_{n=1,2,3...} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh(n\pi b/a)$$

$$V(x)\sin\left(\frac{n'\pi x}{a}\right) = \sum_{n=1,2,3...} C_n \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right)\sinh(n\pi b/a)$$

$$\int_0^a V(x)\sin\left(\frac{n'\pi x}{a}\right)dx = \sum_{n=1,2,3...} C_n \sinh(n\pi b/a)\left[\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right)dx\right]$$

TIP: Since the equation is a standard algebraic equality, whatever we do to one side we must do to the other side!

At this point, the integral on the right is the same integral that appears in the orthogonality statement of the sines (as it should, since that is what we were aiming for in the first place), so that we can replace it with the Kronecker delta times the appropriate constants:

$$\int_0^a V(x)\sin\left(\frac{n'\pi x}{a}\right)dx = \sum_{n=1,2,3...} C_n \sinh(n\pi b/a)\left[\frac{a}{2}\delta_{n,n'}\right]$$

TIP: When applying the orthogonality statement to get the Kronecker delta don't forget the constants that come with it!

Now as we designed, the Kronecker delta collapses the summation symbol, leading to:

$$\int_0^a V(x)\sin\left(\frac{n'\pi x}{a}\right)dx = C_{n'}\sinh(n'\pi b/a)\frac{a}{2}$$

At this point, *n*' is just a label, so we can relabel it everywhere as *n*:

$$\int_0^a V(x)\sin\left(\frac{n\pi x}{a}\right)dx = C_n \sinh(n\pi b/a)\frac{a}{2}$$

Solve for C_n:

$$C_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a V(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

***Step 4*: Insert the Explicit Form of the Complicated Boundary Condition and do the Integral**

Now that we have the general form of the complicated boundary condition, $V(x)$, inside an integral, we can safely expand it into explicit form:

Insert $V(x)=V_0\left(\frac{x}{a}\right)^2$ into $C_n=\frac{2}{a\sinh(n\pi b/a)}\int_0^a V(x)\sin\left(\frac{n\pi x}{a}\right)dx$ to find:

$$C_n=\frac{2}{a\sinh(n\pi b/a)}\int_0^a\left[V_0\left(\frac{x}{a}\right)^2\right]\sin\left(\frac{n\pi x}{a}\right)dx$$

$$C_n=\frac{2V_0}{a^3\sinh(n\pi b/a)}\int_0^a x^2\sin\left(\frac{n\pi x}{a}\right)dx$$

Looking this integral up in a table of integral solutions, I find:

$$C_n=\frac{2V_0}{a^3\sinh(n\pi b/a)}\left[\frac{2xa^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{a}\right)+\left(\frac{2a^3}{n^3\pi^3}-\frac{x^2a}{n\pi}\right)\cos\left(\frac{n\pi x}{a}\right)\right]_0^a$$

$$C_n=\frac{2V_0}{a^3\sinh(n\pi b/a)}$$
$$\times\left(\left[\frac{2(a)a^2}{n^2\pi^2}\sin\left(\frac{n\pi(a)}{a}\right)+\left(\frac{2a^3}{n^3\pi^3}-\frac{(a)^2a}{n\pi}\right)\cos\left(\frac{n\pi(a)}{a}\right)\right]-\left[\frac{2(0)a^2}{n^2\pi^2}\sin\left(\frac{n\pi(0)}{a}\right)+\left(\frac{2a^3}{n^3\pi^3}-\frac{(0)^2a}{n\pi}\right)\cos\left(\frac{n\pi(0)}{a}\right)\right]\right)$$

$$C_n=\frac{2V_0}{a^3\sinh(n\pi b/a)}\left(\left[\left(\frac{2a^3}{n^3\pi^3}-\frac{a^3}{n\pi}\right)(-1)^n\right]-\left[\frac{2a^3}{n^3\pi^3}\right]\right)$$

$$C_n=\frac{2V_0}{n^3\pi^3\sinh(n\pi b/a)}\left((2-n^2\pi^2)(-1)^n-2\right)$$

So that the final solution is:

$$\boxed{\Phi=\frac{2V_0}{\pi^3}\sum_{n=1,2,3...}\frac{(2-n^2\pi^2)(-1)^n-2}{n^3\sinh(n\pi b/a)}\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}$$